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Turbulent magnetic helicity fluxes in solar convective zone

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ABSTRACT

Combined action of helical motions of plasma (the α effect) and non-uniform (differential) rotation is a key dynamo mechanism of solar and galactic large-scale magnetic fields. Dynamics of magnetic helicity of small-scale fields is a crucial mechanism in a nonlinear dynamo saturation where turbulent magnetic helicity fluxes allow to avoid catastrophic quenching of the α effect. The convective zone of the Sun (and solar-like stars as well as galactic discs) are the source for production of turbulent magnetic helicity fluxes. In the framework of the mean-field approach and the spectral τ approximation, we derive turbulent magnetic helicity fluxes using the Coulomb gauge in a density-stratified turbulence. The turbulent magnetic helicity fluxes include non-gradient and gradient contributions. The non-gradient magnetic helicity flux is proportional to a nonlinear effective velocity (which vanishes in the absence of the density stratification) multiplied by small-scale magnetic helicity, while the gradient contributions describe turbulent magnetic diffusion of the small-scale magnetic helicity. In addition, the turbulent magnetic helicity fluxes contain source terms proportional to the kinetic α effect or its gradients, and also contributions caused by the large-scale shear (solar differential rotation). We have demonstrated that the turbulent magnetic helicity fluxes due to the kinetic α effect and its radial derivative in combination with the nonlinear magnetic diffusion of the small-scale magnetic helicity are dominant in the solar convective zone.

Key words: dynamo – MHD – Sun: interior — turbulence – activity

1 INTRODUCTION

The large-scale solar and galactic magnetic fields are generated by a combined action of helical turbulent motions and large-scale differential rotation due to the $\alpha\Omega$ dynamo (see, e.g., Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich et al. 1983; Moffatt & Dormy 2019). A non-zero kinetic helicity produced by a rotating density stratified convective turbulence, causes the kinetic α effect. The dynamo instability is saturated by nonlinear effects. One of the important nonlinear effect is the feedback of the growing large-scale magnetic field on the plasma turbulent motions, so that the turbulent transport coefficients (the α effect, the effective pumping velocity and the turbulent magnetic diffusion) depend on the mean magnetic field $\overline{\boldsymbol{B}}$. The simplest nonlinear saturation mechanism of the dynamo instability is related to the α quenching which prescribes the kinetic α effect to be a decreasing function of the mean magnetic field strength, e.g., $\alpha(\overline{\boldsymbol{B}}) = \alpha_{\rm K} \left(1 + \overline{\boldsymbol{B}}^2/\overline{B}_{\rm eq}^2\right)^{-1}$, where $\alpha_{\rm K} \propto -\tau_0 \, H_{\rm u}$ is the kinetic α effect that is proportional to the kinetic helicity $H_{\rm u} = \langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle$, $\overline{B}_{\rm eq}^2 = 4\pi \, \overline{\rho} \, \langle \boldsymbol{u}^2 \rangle$ is the squared equipartition mean magnetic field, $\boldsymbol{u}$ is the turbulent velocity field, τ_0 is the turbulent time and $\overline{\rho}$ is the mean density. This implies that the mean magnetic field strength at which quenching becomes significant, is estimated from the equipartition between the energy density of the mean magnetic field and the turbulent kinetic energy density. When applied to galactic dynamos, this picture results in robust magnetic field models which are compatible with observations (see, e.g., Ruzmaikin et al. 1988; Shukurov & Subramanian 2021). The above-mention nonlinearity is referred as algebraic nonlinearity.

However this picture is obviously oversimplified and various attempts to suggest a more advanced version of nonlinear dynamo theory have been undertaken (see, e.g., reviews and books by Brandenburg & Subramanian 2005b; Rüdiger et al. 2013; Rincon 2019; Rogachevskii 2021, and references therein). The quantitative theories of the algebraic nonlinearities of the α effect, the turbulent magnetic diffusion and the effective pumping velocity have been developed using the quasi-linear approach for small fluid and magnetic Reynolds numbers (Rüdiger & Kichatinov 1993; Kitchatinov et al. 1994; Rüdiger et al. 2013) and the tau approach for large fluid and magnetic Reynolds numbers (Field et al. 1999; Rogachevskii & Kleeorin 2000, 2001, 2004, 2006).

In addition to the algebraic nonlinearity, there is also a dynamic nonlinearity caused by an evolution of magnetic helicity density of small-scale fields during the nonlinear stage of the mean-field dynamo. In particular, the α effect is the sum of the kinetic and magnetic parts, $\alpha = \alpha_{\rm K} + \alpha_{\rm m}$, where the magnetic α effect, $\alpha_{\rm m} \propto \tau_0 \, H_{\rm c}/(12\pi \, \overline{\rho})$, is proportional to the current helicity $H_{\rm c} = \langle \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle$ of the small-scale magnetic field $\boldsymbol{b}$ (Pouquet et al. 1976). The dynamics of the current helicity H_c is determined by the evolution of the small-scale magnetic helicity density $H_{\rm m} = \langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle$, where magnetic fluctuations $\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{a}$ and $\boldsymbol{a}$ are fluctuations of magnetic vector potential.

Magnetic helicity is fundamental quantity in magnetohydrodynamics and plasma physics (see, e.g., Berger 1999). In particular, the total magnetic helicity, i.e., the sum of the magnetic helicity densities of the large-scale and small-scale magnetic fields, $H_{\rm M} + H_{\rm m}$, integrated over the volume, $\int (H_{\rm M} + H_{\rm m}) \, dr^3$, is conserved for very small microscopic magnetic diffusivity η. Here $H_{\rm M} = \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}}$ is the magnetic helicity density of the large-scale field $\overline{\boldsymbol{B}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{A}}$. Signature of magnetic helicity has been detected in many solar features, including solar active regions (see, e.g., Pevtsov et al. 2014; Zhang et al. 2006, 2012, and references therein).

The governing equation for small-scale magnetic helicity density $H_{\rm m}$ has been derived for an isotropic turbulence by Kleeorin & Ruzmaikin (1982) and for an arbitrary anisotropic turbulence by Kleeorin & Rogachevskii (1999). This equation has been used for analytical study of solar dynamos (Kleeorin et al. 1994, 1995) as well as for mean-field numerical modeling of solar and galactic dynamos (see, e.g., Covas et al. 1997, 1998; Kleeorin et al. 2000, 2002, 2003b,a, 2016; Brandenburg & Subramanian 2005b; Sokoloff et al. 2006; Zhang et al. 2006, 2012; Del Sordo et al. 2013; Safiullin et al. 2018).

As the dynamo amplifies the large-scale magnetic field, the magnetic helicity density $H_{\rm M}$ of the large-scale field grows in time. In particular, the evolution of the large-scale magnetic helicity density $H_{\rm M}$ is determined by the following equation:

$$\frac{\partial H_{\rm M}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}^{(\rm M)} = 2 \boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} - 2\eta H_C \; , \tag{1}$$

where $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$ is the turbulent electromotive force that determines generation and dissipation of the large-scale magnetic field, $2\boldsymbol{\mathcal{E}} \cdot \overline{\mathbf{B}}$ is the source of $H_{\rm M}$ due to the dynamo generated large-scale magnetic field, $\boldsymbol{F}^{(\rm M)}$ is the flux of magnetic helicity density of the large-scale field that determines its transport and $H_{\rm C} = \overline{\boldsymbol{B}} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$ is the current helicity of large-scale field.

Since the total magnetic helicity $\int (H_{\rm M} + H_{\rm m}) \, dr^3$ is conserved, the magnetic helicity density $H_{\rm m}$ of the small-scale field changes during the dynamo action, and its evolution is determined by the dynamic equation (Kleeorin & Ruzmaikin 1982; Zeldovich et al. 1983; Kleeorin et al. 1995; Kleeorin & Rogachevskii 1999):

$$\frac{\partial H_{\rm m}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}^{(\rm m)} = -2 \boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} - 2\eta H_c \; , \tag{2}$$

where $-2\boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}}$ is the source of $H_{\rm m}$ due to the dynamo generated large-scale magnetic field, $\boldsymbol{F}^{(\rm m)}$ is the flux of magnetic helicity density of the small-scale field that determines its transport and $-2\eta H_c$ is the dissipation rate of $H_{\rm m}$. The source of the small-scale and large-scale magnetic helicity densities is only located in turbulent region.

The characteristic decay time of the magnetic helicity density $H_{\rm m}$ of the small-scale field is of the order of $T_{\rm m} = \tau_0 \, {\rm Rm}$, while the characteristic time for the decay of kinetic helicity is of the order of the turn-over time $\tau_0 = \ell_0 / u_0$ of turbulent eddies in the integral turbulence scale ℓ_0, where ${\rm Rm} = \ell_0 \, u_0 / \eta$ is the magnetic Reynolds number. The current helicity $H_{\rm c}$ of the small-scale field is not an integral of motion and the characteristic decay time of $H_{\rm c}$ varies from a short timescale τ_0 to much larger timescales. On the other hand, the characteristic decay times of the current helicity of large-scale field, $H_{\rm C}$, and of the large-scale magnetic helicity $H_{\rm M}$ are of the order of the turbulent diffusion time. For weakly inhomogeneous turbulence the current helicity density $H_{\rm c}$ of the small-scale field is proportional to the small-scale magnetic helicity density $H_{\rm m}$ (Kleeorin & Rogachevskii 1999).

Using the steady-state solution of Eq. (2) with a zero turbulent flux $\boldsymbol{F}^{(\rm m)} = 0$ of magnetic helicity density of small-scale field and a zero current helicity of large-scale field, $H_{\rm C}$, it has been concluded that the critical mean magnetic field strength, $\overline{B}_{\rm cr}$, at which the dynamic α quenching becomes significant, in fact is much lower than the equipartition value, e.g. $\overline{B}_{\rm cr} = \overline{B}_{\rm eq} \, {\rm Rm}^{-1/2}$ (Vainshtein & Cattaneo 1992; Gruzinov & Diamond 1994). In astrophysics, e.g., in galactic disks and in the convective zone of the sun, magnetic Reynolds numbers are very large. Therefore, for large magnetic Reynolds numbers the dynamo action should saturate at a magnetic field strength that is much lower than the equipartition value. This effect is referred as to a catastrophic quenching of the α effect (Vainshtein & Cattaneo 1992; Gruzinov & Diamond 1994). On the other hand, the observed large-scale field strengths in spiral galaxies is of the order of the equipartition value (see, e.g., Ruzmaikin et al. 1988; Shukurov & Subramanian 2021), and the observed solar and stellar magnetic fields are much larger than $\overline{B}_{\rm cr}$ (see, e.g., Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich et al. 1983).

The evolution of magnetic helicity appears however to be a more complicated process than can simply be described by a balance of magnetic helicity in a given volume. It is necessary to take into account fluxes of magnetic helicity (Kleeorin et al. 2000). This implies that the turbulent transport of magnetic helicity through the boundaries (the open boundary conditions in simulations) should be taken into account (Blackman & Field 2000). Different forms of magnetic helicity fluxes have been suggested in various studies (Covas et al. 1997, 1998; Kleeorin & Rogachevskii 1999; Kleeorin et al. 2000, 2002; Vishniac & Cho 2001; Subramanian & Brandenburg 2004; Brandenburg & Subramanian 2005b). Turbulent fluxes of small-scale magnetic helicity fluxes have been measured in numerical simulations (Käpylä et al. 2010; Mitra et al. 2010; Hubbard & Brandenburg 2010, 2011, 2012; Del Sordo et al. 2013), and in solar observations (Chae et al. 2001; Pariat et al. 2005; Pevtsov et al. 2014; Hawkes & Berger 2018).

Taking into account turbulent fluxes of the small-scale magnetic helicity, it has been shown by numerical simulations that a nonlinear galactic dynamo governed by a dynamic equation for the magnetic helicity density $H_{\rm m}$ of small-scale field results in a steady mean magnetic field comparable with the equipartition magnetic field (see, e.g., Kleeorin et al. 2000, 2002, 2003b,a; Blackman & Brandenburg 2002; Brandenburg & Subramanian 2005b; Shukurov et al. 2006; Del Sordo et al. 2013). Numerical simulations demonstrate that the dynamics of the small-scale magnetic helicity in the presence of the turbulent magnetic helicity fluxes play a crucial role in the solar dynamo as well (see, e.g., Kleeorin et al. 2003b, 2016, 2020; Sokoloff et al. 2006; Zhang et al. 2006, 2012; Käpylä et al. 2010; Guerrero et al. 2010; Hubbard & Brandenburg 2012; Del Sordo et al. 2013; Safiullin et al. 2018; Rincon 2021).

Due to very important role of the turbulent magnetic helicity fluxes in nonlinear dynamos, in the present study we perform a rigorous derivation of these fluxes applying the mean-field theory, adopting the Coulomb gauge and considering a strongly density-stratified turbulence. We show that the turbulent magnetic helicity fluxes contain non-gradient and gradient contributions. The non-gradient magnetic helicity fluxes are product of a nonlinear effective velocity and small-scale magnetic helicity. The gradient contributions determine a nonlinear magnetic diffusion of the small-scale magnetic helicity. We also demonstrate that the turbulent magnetic helicity fluxes include source terms proportional to the kinetic α effect or its gradients. In the present study we do not consider an algebraic quenching of the turbulent magnetic helicity fluxes that is a subject of a separate study.

This paper is organized as follows. In Section 2, we derive equation for the magnetic helicity of small-scale fields which includes divergence of the turbulent magnetic helicity flux. In Section 3 we discuss the results of calculations of the turbulent flux of magnetic helicity of the small-scale fields. In addition, we obtain a general form of turbulent flux of the magnetic helicity using symmetry arguments. In Section 4, we consider the turbulent magnetic helicity flux in the solar convective zone. Finally, in Section 5, we discuss our results and draw conclusions. In Appendixes A and B we discuss approximations and procedure of the derivation of turbulent flux of magnetic helicity. In Appendix C we determine the effect of large-scale shear on turbulent flux of the magnetic helicity. Applying the method described in Appendixes A–C, we determine various contributions to the turbulent flux of the small-scale magnetic helicity in Appendix D. In particular, we present the general form of turbulent transport coefficients entering in the turbulent flux of the small-scale magnetic helicity. For better understanding of the physics related to various contributions to the turbulent flux of the small-scale magnetic helicity, in Appendix E we consider a more simple case with a large-scale linear velocity shear and present turbulent transport coefficients in the Cartesian coordinates.

2 EQUATION FOR THE MAGNETIC HELICITY

In this Section, we derive an equation for the small-scale magnetic helicity. The induction equation for fluctuations of magnetic field $\boldsymbol{b}$ reads

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{\nabla} \times \Big[\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \overline{\boldsymbol{B}} + \boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle - \eta \boldsymbol{\nabla} \times \boldsymbol{b}\Big], \tag{3}$$

where in the framework of the mean-field approach, we separate magnetic and velocity fields into mean and fluctuations, $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$ and $\overline{\boldsymbol{B}} = \langle \boldsymbol{B} \rangle$ is the mean magnetic field, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, and $\overline{\boldsymbol{U}} = \langle \boldsymbol{U} \rangle$ is the mean fluid velocity describing, e.g., the differential rotation, η is the magnetic diffusion due to electrical conductivity of fluid. The equation for magnetic fluctuations is obtained by subtracting induction equation for the the mean magnetic field $\overline{\boldsymbol{B}}$ from that for the total field $\boldsymbol{B}(t, \boldsymbol{x})$. The equation for fluctuations of the vector potential $\boldsymbol{a}$ follows from induction equation (3)

$$\frac{\partial \boldsymbol{a}}{\partial t} = \overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \overline{\boldsymbol{B}} + \boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle - \eta \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla} \phi, \tag{4}$$

where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{A} = \overline{\boldsymbol{A}} + \boldsymbol{a}$, and $\overline{\boldsymbol{A}} = \langle \boldsymbol{A} \rangle$ is the mean vector potential, $\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{a}$ and ϕ are fluctuations of the scalar potential. We multiply Eq. (3) by $\boldsymbol{a}$ and Eq. (4) by $\boldsymbol{b}$, add them and average over an ensemble of turbulent fields. This yields an equation for the magnetic helicity $H_{\rm m} = \langle \boldsymbol{a}(\mathbf{x}) \cdot \boldsymbol{b}(\mathbf{x}) \rangle$ of the small-scale fields as

$$\frac{\partial H_{\rm m}}{\partial t} = -2 \boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} - 2\eta \langle \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle - \boldsymbol{\nabla} \cdot \boldsymbol{F}^{(\rm m)}, \tag{5}$$

where $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$ is the turbulent electromotive force, and the turbulent flux of magnetic helicity $\boldsymbol{F}^{(\rm m)}$ of the small-scale fields is given by

$$\boldsymbol{F}^{(\rm m)} = \overline{\boldsymbol{U}}\, H_{\rm m} - \big\langle \boldsymbol{b}\, (\boldsymbol{a} \cdot \overline{\boldsymbol{U}}) \big\rangle + \big\langle \boldsymbol{u}\, (\boldsymbol{a} \cdot \overline{\boldsymbol{B}}) \big\rangle - \overline{\boldsymbol{B}}\, \langle \boldsymbol{a} \cdot \boldsymbol{u} \rangle - \eta\, \langle \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle + \langle \boldsymbol{a} \times (\boldsymbol{u} \times \boldsymbol{b}) \rangle - \langle \boldsymbol{b}\, \phi \rangle\,. \tag{6}$$

Using the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{a} = 0$, we obtain that $\boldsymbol{\nabla} \times \boldsymbol{b} = -\Delta \boldsymbol{a}$ and $\boldsymbol{a} = -\Delta^{-1}\, \boldsymbol{\nabla} \times \boldsymbol{b}$. The Coulomb gauge also allows us to find fluctuations of the scalar potential ϕ. Indeed, equation for $\boldsymbol{\nabla} \cdot \boldsymbol{a}$ which follows from Eq. (4), yields expression for fluctuations of the scalar potential ϕ, so that the correlation function $\langle b_i\, \phi \rangle$ reads

$$\langle b_i\, \phi \rangle = \langle b_i\, a_j \rangle\, \overline{U}_j - \big\langle b_i\, \Delta^{-1}\, (\boldsymbol{\nabla} \times \boldsymbol{u})_j \big\rangle\, \overline{B}_j - \big\langle b_i\, \Delta^{-1}\, b_j \big\rangle\, \overline{W}_j + \big\langle b_i\, \Delta^{-1}\, u_j \big\rangle\, (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})_j - \big\langle b_i\, \Delta^{-1}\, \boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{b}) \big\rangle\,. \tag{7}$$

where $\overline{\boldsymbol{W}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$ is the mean vorticity and $\langle b_i\, a_j \rangle = -\big\langle b_i\, \Delta^{-1}\, (\boldsymbol{\nabla} \times \boldsymbol{b})_j \big\rangle$. Equations (6)–(7) yield the turbulent flux of magnetic helicity $\boldsymbol{F}^{(\rm m)}$ of the small-scale fields as

$$F_i^{(\rm m)} = \overline{U}_i\, H_{\rm m} + \overline{W}_j\, \big\langle b_i\, \Delta^{-1}\, b_j \big\rangle + \overline{B}_j\, \langle u_i\, a_j \rangle - \overline{B}_i\, \langle u_j\, a_j \rangle + \overline{B}_j\, \big\langle b_i\, \Delta^{-1}\, (\boldsymbol{\nabla} \times \boldsymbol{u})_j \big\rangle + F_i^{(\eta)} - (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})_j\, \big\langle b_i\, \Delta^{-1}\, u_j \big\rangle + F_i^{(\rm III)}, \tag{8}$$

where $\langle u_i\, a_j \rangle = -\big\langle u_i\, \Delta^{-1}\, (\boldsymbol{\nabla} \times \boldsymbol{b})_j \big\rangle$, $\boldsymbol{F}^{(\eta)} = -\eta\, \langle \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle$ is the flux caused by the microscopic magnetic diffusion η and $\boldsymbol{F}^{(\rm III)}$ is the flux that is determined by the third-order moments, is given by

$$\boldsymbol{F}^{(\rm III)} = \big\langle \boldsymbol{b}\, \Delta^{-1}\, \boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{b}) \big\rangle + \langle \boldsymbol{a} \times (\boldsymbol{u} \times \boldsymbol{b}) \rangle\,. \tag{9}$$

Equations (5)–(9) are exact equations. Note that only in the Coulomb gauge, the scalar potential ϕ is described by the stationary equation. For all other gauge conditions, the scalar potential ϕ is determined by a non-stationary equation. Also for the Coulomb gauge the relation between the magnetic α effect and small-scale magnetic helicity is most simple.

3 GENERAL FORM OF TURBULENT FLUX OF THE MAGNETIC HELICITY

In this Section we discuss the results of calculations of the turbulent flux of magnetic helicity of the small-scale fields. General form of turbulent flux $\boldsymbol{F}^{(\mathrm{m})}$ of the magnetic helicity can be obtained from symmetry reasoning. Indeed, the turbulent flux $\boldsymbol{F}^{(\mathrm{m})}$ is the pseudo-vector which should contain two pseudo-scalars: the magnetic helicity, H_m, and the kinetic α effect, α_K, and their first spatial derivatives. In addition, the contributions $F_i^{(\mathrm{S0})}$ to the turbulent magnetic helicity flux caused by the large-scale shear (differential rotation) should contain the the pseudo-vector $\overline{\boldsymbol{W}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$, where $\overline{\boldsymbol{U}} = \boldsymbol{\delta}\Omega \times \boldsymbol{r}$ is the large-scale velocity describing the differential rotation $\boldsymbol{\delta}\Omega$.

All turbulent transport coefficients entering in the turbulent flux $\boldsymbol{F}^{(\mathrm{m})}$ of magnetic helicity of the small-scale fields should be quadratic in the large-scale magnetic field $\overline{\boldsymbol{B}}$, i.e., they should be proportional to $\overline{B}^2$ or $\overline{V}_\mathrm{A}^2 = \overline{B}^2/(4\pi\overline{\rho})$, where $\overline{\rho}$ is the mean plasma density and $\overline{V}_\mathrm{A}$ is the mean Alfvén speed. On the other hand, the turbulent flux $\boldsymbol{F}^{(\mathrm{m})}$ of the magnetic helicity should vanish in the absence of turbulence. This implies that all turbulent transport coefficients entering in the turbulent flux $\boldsymbol{F}^{(\mathrm{m})}$ should be proportional to turbulent correlation time τ_0 or turbulent integral scale ℓ_0. Some of the turbulent transport coefficients are caused by the plasma density stratification, i.e., they are proportional to $\boldsymbol{\lambda} = -\boldsymbol{\nabla} \ln \overline{\rho}$.

Using the theoretical approach based on the spectral τ approximation which is valid for large fluid and magnetic Reynolds numbers, and the multi-scale approach, we obtain the turbulent flux of the small-scale magnetic helicity as

$$F_i^{(\mathrm{m})} = \left(\overline{U}_i + V_i^{(\mathrm{H})}\right) H_\mathrm{m} - D_{ij}^{(\mathrm{H})} \nabla_j H_\mathrm{m} + N_i^{(\alpha)} \alpha_\mathrm{K} + M_{ij}^{(\alpha)} \nabla_j \alpha_\mathrm{K} + F_i^{(\mathrm{S0})}, \tag{10}$$

where $\alpha_\mathrm{K} = -\tau_0 H_\mathrm{u}/3$ is the kinetic α effect. Details of the derivation of Eq. (10) are described in Appendixes A–C. The general form of the turbulent transport coefficients entering in the turbulent flux (10) of magnetic helicity of the small-scale fields is given by Eqs. (D2)–(D6) in Appendix D. These turbulent transport coefficients of the turbulent magnetic helicity flux in spherical coordinates are given in the next section and in the Cartesian coordinates are discussed in Appendix E.

The turbulent flux of the small-scale magnetic helicity includes the non-gradient and gradient contributions. The non-gradient contribution to the turbulent flux of magnetic helicity is proportional to the sum of the mean velocity $\overline{\boldsymbol{U}} = \boldsymbol{\delta}\Omega \times \boldsymbol{r}$ and the turbulent pumping velocity $\boldsymbol{V}^{(\mathrm{H})}$ which is multiplied by small-scale magnetic helicity H_m, while the gradient contribution $-D_{ij}^{(\mathrm{H})} \nabla_j H_\mathrm{m}$ describe the turbulent magnetic diffusion of the small-scale magnetic helicity. The effective pumping velocity of the small-scale magnetic helicity $\boldsymbol{V}^{(\mathrm{H})}$ vanishes in the absence of the density stratification. In addition, the turbulent magnetic helicity flux contains the source term $\boldsymbol{N}^{(\alpha)} \alpha_\mathrm{K}$ proportional to the kinetic α effect, and the source term $-M_{ij}^{(\alpha)} \nabla_j \alpha_\mathrm{K}$ proportional to the gradient $\nabla_j \alpha_\mathrm{K}$ of the kinetic α effect. The turbulent magnetic helicity flux also have contributions caused by the large-scale shear (differential rotation) in the turbulent flow.

We assume that the turbulent flux of the magnetic helicity $\boldsymbol{F}^{(\mathrm{III})}$ containing the third-order moments [see equation (9)], is determined using the turbulent diffusion approximation as $\boldsymbol{F}^{(\mathrm{III})} = -D_T^{(\mathrm{H})} \boldsymbol{\nabla} H_\mathrm{m}$. The contribution to the turbulent magnetic helicity flux, $-D_T^{(\mathrm{H})} \boldsymbol{\nabla} H_\mathrm{m}$, caused by the turbulent diffusion, has been used in mean-field numerical simulations by Covas et al. (1997, 1998); Kleeorin et al. (2002, 2003a).

The turbulent diffusion of the small-scale magnetic helicity can be interpreted as follows. The random flows existing in the interstellar medium consist of a combination of small-scale motions, which are affected by magnetic forces resulting in a steady-state of the dynamo, and a micro-turbulence which is supported by a strong random driver (e.g., supernovae explosions which can be considered as independent of the galactic magnetic field). The large-scale magnetic field is smoothed over both kinds of turbulent fluctuations, while the small-scale magnetic field is smoothed over micro-turbulent fluctuations only. It is the smoothing over the micro-turbulent fluctuations that gives the coefficient $D_T^{(\mathrm{H})} = C_D \eta_{_T}$ with a free dimensionless constant $C_D \sim 0.1$. Here $\eta_{_T}$ is the turbulent diffusion coefficient of the mean magnetic field.

The magnetic helicity flux $\boldsymbol{F}^{(\eta)} = -\eta \langle \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle$ due to the microscopic magnetic diffusion η is given by $\boldsymbol{F}^{(\eta)} = -\frac{1}{3}\eta \boldsymbol{\nabla} H_\mathrm{m}$. This flux in astrophysical systems is very small and neglected here.

4 TURBULENT MAGNETIC HELICITY FLUX IN THE SOLAR CONVECTIVE ZONE

In this Section we discuss the results of calculations of the turbulent magnetic helicity flux in the solar convective zone, where we use spherical coordinates (r, ϑ, φ). The radial turbulent flux of the small-scale magnetic helicity is given by

$$F_r^{(\mathrm{m})} = V_r^{(\mathrm{H})} H_\mathrm{m} - D_{rj}^{(\mathrm{H})} \nabla_j H_\mathrm{m} + N_r^{(\alpha)} \alpha_\mathrm{K} + M_{rj}^{(\alpha)} \nabla_j \alpha_\mathrm{K} + F_r^{(\mathrm{S0})}. \tag{11}$$

The general forms of the turbulent transport coefficients entering in the turbulent flux $\boldsymbol{F}^{(\mathrm{m})}$ of magnetic helicity of the small-scale fields are given by Eqs. (D2)–(D6) in Appendix D. In view of applications to the solar convective zone, the turbulent transport coefficients of the turbulent magnetic helicity flux in spherical coordinates are specified below:

$$V_r^{(\mathrm{H})} = -\frac{1}{15}\tau_0 \overline{V}_\mathrm{A}^2 \lambda \left[1 + 7\beta_r^2 - \frac{173}{14} \sin\vartheta\, \tau_0\, \delta\Omega\, \beta_r \beta_\varphi\right], \tag{12}$$

$$D_{rr}^{(\mathrm{H})} = D_T^{(\mathrm{H})} + \frac{1}{30}\tau_0 \overline{V}_\mathrm{A}^2 \left(5 - 4\beta_r^2\right), \tag{13}$$

$$D_{r\vartheta}^{(\mathrm{H})} = \frac{2(80 + 17q)}{105}\tau_0^2 \overline{V}_\mathrm{A}^2\, \delta\Omega\, \beta_r\, \beta_\varphi \cos\vartheta, \tag{14}$$

$$
\begin{aligned}
N_r^{(\alpha)} &= -\frac{1}{10}\ell_0^2\,\overline{B}^2\,\lambda\left[1+\frac{7q-2}{q}\,\beta_r^2\right.\\
&\left.-\frac{216(q-1)}{7(3q-1)}\,\tau_0\,\delta\Omega\,\beta_r\,\beta_\varphi\,\sin\vartheta\right], \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
M_{rr}^{(\alpha)} &= \frac{2q-1}{20q}\ell_0^2\,\overline{B}^2\left[1+\frac{20q-23}{2q-1}\,\beta_r^2\right.\\
&\left.-\frac{32q(q-1)}{(2q-1)\,(3q-1)}\,\tau_0\,\delta\Omega\,\beta_r\,\beta_\varphi\,\sin\vartheta\right], \qquad (16)
\end{aligned}
$$

$$
M_{r\vartheta}^{(\alpha)} = \frac{8(q-1)}{3q-1}\ell_0^2\,\overline{B}^2\,\tau_0\,\delta\Omega\,\beta_r\,\beta_\varphi\,\cos\vartheta, \qquad (17)
$$

$$
\begin{aligned}
F_r^{(\mathrm{S0})} &= -\frac{2}{9}\,\delta\Omega\,\cos\vartheta\left\{4\,\ell_0^2\,\overline{B}_r^2+\left[\frac{\overline{V}_\mathrm{A}^2}{\langle\boldsymbol{u}^2\rangle}\left(1-\frac{3}{11}\beta_r^2\right)\right.\right.\\
&\left.\left.+\frac{3(q-1)}{q+1}\right]\ell_b^2\,\langle\boldsymbol{b}^2\rangle\right\}, \qquad (18)
\end{aligned}
$$

where $\boldsymbol{\beta} = \overline{\boldsymbol{B}}/\overline{B}$ is the unit vector along the mean magnetic field, $\overline{\boldsymbol{U}} = \delta\Omega\, r\,\sin\vartheta\,\boldsymbol{e}_\varphi$ is the mean velocity caused by the differential rotation $\delta\Omega = \Omega(r,\vartheta) - \Omega(r = R_\odot, \vartheta)$. Here $\Omega(r = R_\odot, \vartheta) = \Omega_0(1 - C_2\cos^2\vartheta - C_4\cos^4\vartheta)$ with $\Omega_0 = 2.83\times10^{-6}$ s^{-1}, $C_2 = 0.121$ and $C_4 = 0.173$ (LaBonte & Howard 1982), $R_\odot$ is the solar radius, $\boldsymbol{\lambda} = \lambda\,\boldsymbol{e}_r$, ℓ_b is the energy containing scale of magnetic fluctuations with a zero mean magnetic field and q is the exponent the spectrum of the turbulent kinetic energy (the exponent $q = 5/3$ corresponds to the Kolmogorov spectrum of the turbulent kinetic energy).

In derivation of Eqs. (12)–(18), we take into account that for weakly inhomogeneous turbulence $H_\mathrm{c} \approx H_\mathrm{m}/\ell_0^2$, and we neglect small terms $\sim \mathrm{O}[\ell_0^2/L_\mathrm{m}^2]$ with L_m being characteristic scale of spatial variations of H_m. We neglect also small contributions proportional to spatial derivatives of the mean magnetic field, and spatial derivatives of $\langle\boldsymbol{u}^2\rangle$ and $\delta\Omega$.

Let us discuss the obtained results. For illustration, in Fig. 1 we show the radial profile of the total angular velocity $\Omega(r)/\Omega_\odot$ in the solar convective zone that includes the uniform and differential rotation specified for the latitude $\phi_* = 30°$. The theoretical profile (solid line) of the total angular velocity (Rogachevskii & Kleeorin 2018) is compared with the radial profile of the solar angular velocity (stars) obtained from the helioseismology observational data (Kosovichev et al. 1997) specified for the latitude $\phi = 30°$ and normalized by the solar rotation frequency $\Omega_\odot(\phi_* = 0)$ at the equator, where $\Omega/\Omega_\odot$ is given by Eq. (3.14) derived by Rogachevskii & Kleeorin (2018). In Figs. 1–2 we also show the radial profile of the kinetic α effect, $\alpha_\mathrm{K}/\alpha_\mathrm{max}$ which is specified for the latitude $\phi = 30°$ and given by Eq. (22) derived by Kleeorin & Rogachevskii (2003).

In the upper part of the solar convective zone for the latitude $\phi_* > 0$ (the Northern Hemisphere), the kinetic α effect is positive, $\alpha_\mathrm{K} > 0$ (see Fig. 2). On the other hand, the magnetic α effect in this region is negative, i.e., $\alpha_\mathrm{M} = \tau_0\,H_\mathrm{c}/(4\pi\overline{\rho}) < 0$. This implies that the current helicity $H_\mathrm{c} < 0$ as well as the magnetic helicity $H_\mathrm{m} < 0$ are negative the Northern Hemisphere. Here for simplicity, we choose the radial profile of the poloidal and toroidal field as $\overline{B}_r = \overline{B}_{r0}\,\sin[\pi(r - 0.73R_\odot)/(0.6R_\odot)]$ and $\overline{B}_\varphi = \overline{B}_{\varphi0}\,\cos[\pi(r - 0.73R_\odot)/(0.6R_\odot)]$, where $\overline{B}_{r0}$ is the surface mean magnetic field measured in Gauss. To avoid catastrophic quenching, the radial component of the turbulent flux of the small-scale magnetic helicity $F_r^{(\mathrm{m})} < 0$ should be negative for the Northern Hemisphere.

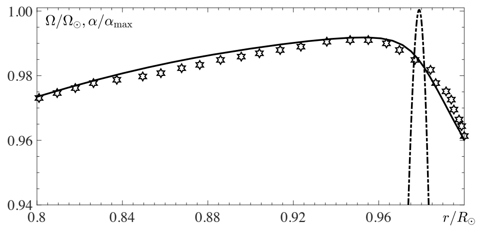

Figure 1. The theoretical radial profiles of the total angular velocity $\Omega(r)/\Omega_\odot$ (solid) that includes the uniform and differential rotation specified for the latitude $\phi_* = 30°$ and the normalized kinetic α effect, $\alpha_\mathrm{K}/\alpha_\mathrm{max}$ (dashed). The theoretical profile of the total angular velocity is compared with the radial profile of the solar angular velocity obtained from the helioseismology observational data (stars) specified for the latitude $\phi_* = 30°$ and normalized by the solar rotation frequency $\Omega_\odot(\phi_* = 0)$ at the equator (Kosovichev et al. 1997), where $R_\odot$ is the solar radius. The profile $\alpha_\mathrm{K}(r) \equiv \alpha_{\varphi\varphi}^{(\mathrm{K})}$ is given by Eq. (22) derived by Kleeorin & Rogachevskii (2003), and $\Omega(r)/\Omega_\odot$ is given by Eq. (3.14) derived by Rogachevskii & Kleeorin (2018).

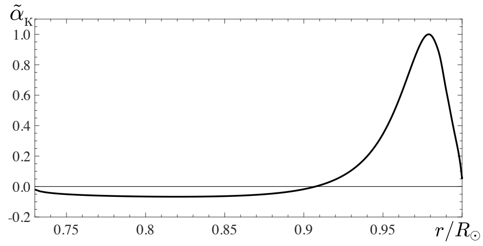

Figure 2. The radial profile of the normalized kinetic α effect, $\tilde{\alpha}_\mathrm{K} = \alpha_\mathrm{K}/\alpha_\mathrm{max}$, specified for the latitude $\phi_* = 30°$ and given by Eq. (22) derived by Kleeorin & Rogachevskii (2003).

In Figs. 3 and 4 we show the radial profiles of the effective pumping velocity $V_r^{(\mathrm{H})}(r)$ and turbulent diffusion $D_{rr}^{(\mathrm{H})}(r)$ of the small-scale magnetic helicity. In Figs. 5 and 6 we plot the radial profiles of the turbulent magnetic helicity fluxes caused by the source terms $F_1^{(\alpha)}(r) = N_r^{(\alpha)}\,\alpha_\mathrm{K}$ and $F_2^{(\alpha)}(r) = M_{rr}^{(\alpha)}\,\nabla_r\alpha_\mathrm{K}$, which are proportional to the kinetic α effect and its radial derivative, as well as their sum $F_r^{(\alpha)}(r) = N_r^{(\alpha)}\,\alpha_\mathrm{K} + M_{rr}^{(\alpha)}\,\nabla_r\alpha_\mathrm{K}$. In Fig. 6 we also show the contribution $\boldsymbol{F}^{(\mathrm{S0})}(r)$ to the turbulent magnetic helicity flux caused by the large-scale shear (differential rotation). Finally, in Fig. 7 we plot the radial profile of the total source flux of the magnetic helicity $F_\mathrm{tot}(r) = N_r^{(\alpha)}\,\alpha_\mathrm{K} + M_{rr}^{(\alpha)}\,\nabla_r\alpha_\mathrm{K} + F_r^{(\mathrm{S0})}$ that is independent of the magnetic helicity and its radial derivative.

As follows from Figs. 3–7 as well as Eqs. (11)–(18), the negative contribution to the turbulent magnetic helicity flux $F_r^{(\mathrm{m})}$ in the range of the generation of the mean magnetic field, is due to the source flux $F_r^{(\alpha)} = N_r^{(\alpha)}\,\alpha_\mathrm{K} + M_{rr}^{(\alpha)}\,\nabla_r\alpha_\mathrm{K}$, and the contribution $\boldsymbol{F}^{(\mathrm{S0})}$ to the turbulent magnetic helicity flux caused by the large-scale shear (differential rotation). Here we take into account that $\delta\Omega > 0$ at $0.8 <$

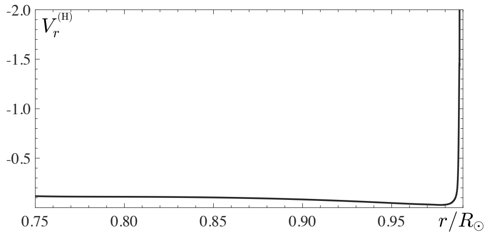

Figure 3. The radial profile of the effective pumping velocity $V_r^{(\mathrm{H})}$ of the small-scale magnetic helicity given by Eq. (12), and measured in m s^{-1}.

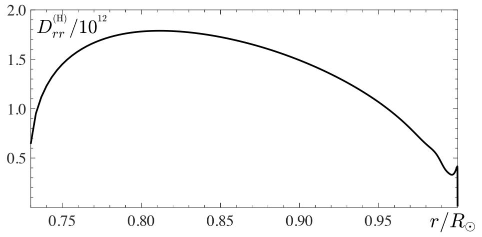

Figure 4. The radial profile of turbulent diffusion $D_{rr}^{(\mathrm{H})}(r)$ of the small-scale magnetic helicity given by Eq. (13) and measured in cm^2 s^{-1}.

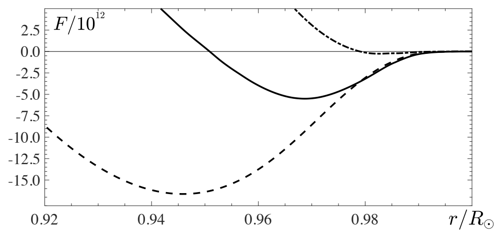

Figure 5. The radial profile of the turbulent magnetic helicity fluxes caused by the source terms $F_1^{(\alpha)} = N_r^{(\alpha)}\,\alpha_{\rm K}$ (dashed) and $F_2^{(\alpha)} = M_{rr}^{(\alpha)}\,\nabla_r\alpha_{\rm K}$ (dashed-dotted) which are proportional to the kinetic α effect and its radial derivative, as well as their sum $F_r^{(\alpha)} = N_r^{(\alpha)}\,\alpha_{\rm K} + M_{rr}^{(\alpha)}\,\nabla_r\alpha_{\rm K}$ (solid), where $N_r^{(\alpha)}$ and $M_{rr}^{(\alpha)}$ are given by Eqs. (15) and (16), respectively. The fluxes are specified for the latitude $\phi_\ast = 30^\circ$ and measured in G^2 cm^2 s^{-1}.

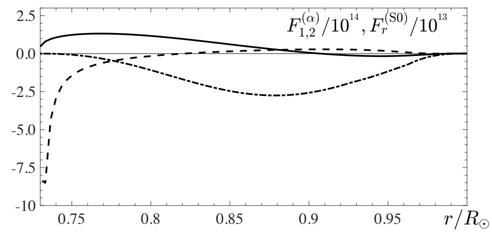

Figure 6. The radial profiles of the turbulent magnetic helicity fluxes caused by the source terms $F_1^{(\alpha)} = N_r^{(\alpha)}\,\alpha_{\rm K}$ (solid), $F_2^{(\alpha)} = M_{rr}^{(\alpha)}\,\nabla_r\alpha_{\rm K}$ (dashed) and the contribution $F_r^{(\mathrm{S0})}$ (dashed-dotted) to the turbulent magnetic helicity flux caused by the large-scale shear (differential rotation) , where $N_r^{(\alpha)}$, $M_{rr}^{(\alpha)}$ and $F_r^{(\mathrm{S0})}$ are given by Eqs. (15), (16) and (18), respectively. The fluxes are specified for the latitude $\phi_\ast = 30^\circ$ and measured in G^2 cm^2 s^{-1}.

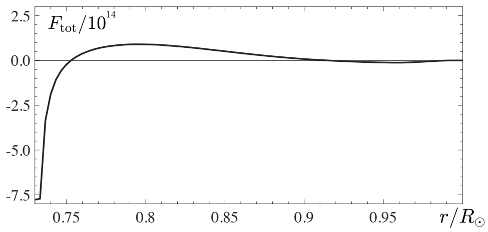

Figure 7. The radial profile of the total source flux $F_{\rm tot} = N_r^{(\alpha)}\,\alpha_{\rm K} + M_{rr}^{(\alpha)}\,\nabla_r\alpha_{\rm K} + F_r^{(\mathrm{S0})}$ of the magnetic helicity that is independent of the magnetic helicity and its radial derivative. Here the flux is measured in G^2 cm^2 s^{-1}.

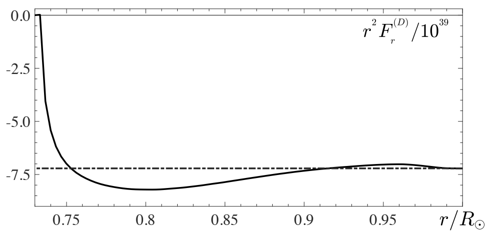

Figure 8. Turbulent diffusion flux $r^2 F_r^{(D)}$ (solid line) and the flux $r^2\,[F_r^{(D)}(r) + F_{\rm tot}(r)]$ (dashed-dotted line) of magnetic helicity per unit solid angle which are measured in Mx^2 h^{-1}.

$r/R_\odot < 1$ (see Fig. 1), where the differential rotation $\delta\Omega = \Omega(r) - \Omega(r = R_\odot)$.

The small-scale magnetic helicity is not accumulated inside the solar convective zone due to turbulent magnetic diffusion flux, $F_r^{(D)}$. In Fig. 8 we show the turbulent diffusion flux $r^2 F_r^{(D)}$ (solid line) of magnetic helicity per unit solid angle and the flux $[F_r^{(D)}(r) + F_{\rm tot}(r)]\,r^2$ (dashed-dotted line) of magnetic helicity per unit solid angle which are measured in Mx^2 h^{-1}. As follows from Fig. 8, the flux $[F_r^{(D)}(r) + F_{\rm tot}(r)]\,r^2$ (the sum of the turbulent diffusion flux and total source flux of magnetic helicity) of small-scale field per unit solid angle is independent of r, i.e.,

$$[F_r^{(D)}(r) + F_{\rm tot}(r)]\,r^2 \approx F_{\rm tot}(r = 0.73 R_\odot)\,(0.73 R_\odot)^2. \quad (19)$$

Here we take into account that the turbulent diffusion flux $F_r^{(D)}(r = 0.73 R_\odot) \to 0$ vanishes at the bottom of the convective zone, $r = 0.73 R_\odot$, where the turbulence intensity vanishes (see Fig. 8). Equation (19) implies that there is no accumulation of small-scale magnetic helicity inside the solar convective zone.

In Fig. 9 we compare the theoretical predictions for flux $\Phi_D \equiv F_r^{(D)}(r = R_\odot)\,R_\odot^2\,\delta\phi_\ast$ with the observational values of Φ_D which are taken from Fig. 8a by Chae et al. (2001), where time variations of the rates of magnetic helicity change by photospheric motions (which do not include differential rotation) are shown. Here the flux Φ_D is measured in Mx^2 h^{-1} and $\delta\phi_\ast = 2\pi\,\sin(\pi/4)$ is the solid angle corresponding to the thickness of the Royal sunspot region. The theoretical values for Φ_D are given for different values of the mean magnetic field, $\overline{B}_{\rm bot}$ and $\overline{B}_{\rm top}$, at the bottom and top of the solar convective zone (see the caption of Fig. 9). Note that the measurements of the magnetic helicity flux are based on the equation $\partial H_{\rm m}/\partial t = -2\oint(\boldsymbol{u}\cdot\boldsymbol{a}_p)\,\boldsymbol{b}_z\,dS$

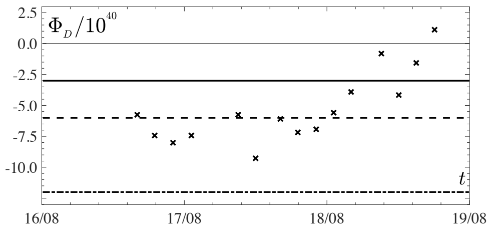

Figure 9. Comparison of the theoretical predictions for $\Phi_D = F_r^{(D)}(r = R_\odot)\, R_\odot^2\, \delta\phi_*$ with the observational values of Φ_D (slanting crosses) which are taken from Fig. 8a by Chae et al. (2001), where time variations of the rates of magnetic helicity change by photospheric motions (which do not include differential rotation) are shown. Here the flux Φ_D is measured in Mx^2 h^{-1} and $\delta\phi_* = 2\pi \sin(\pi/4)$ is the solid angle corresponding to the thickness of the Royal sunspot region. The theoretical values for Φ_D are given for different values of the mean magnetic field, $\overline{B}_{\rm bot}$ and $\overline{B}_{\rm top}$, at the bottom and top of the solar convective zone (i.e., thick solid line is for $\overline{B}_{\rm bot} = 10^3$ G and $\overline{B}_{\rm top} = 8$ G; dashed line is for $\overline{B}_{\rm bot} = 1.4 \times 10^3$ G and $\overline{B}_{\rm top} = 11$ G and dashed-dotted line is for $\overline{B}_{\rm bot} = 2 \times 10^3$ G and $\overline{B}_{\rm top} = 16$ G).

(Chae et al. 2001; Pevtsov et al. 2014), where we use here the lower-case letters for the small-scale fields. This implies that the measurements by Chae et al. (2001) are based on the calculation of the third-order moment, $\langle (\boldsymbol{u} \cdot \boldsymbol{a}_p)\, \boldsymbol{b}_z \rangle$, which we describe using the turbulent diffusion approximation, $F_r^{(D)} = -D_{rr}^{(\rm H)}\, \nabla_r H_{\rm m}$. As follows from Fig. 9, the theoretical predictions for flux Φ_D are in agreement with the observational values of Φ_D.

5 DISCUSSION AND CONCLUSIONS

In the present study, turbulent magnetic helicity fluxes of small-scale field are derived applying the mean-field approach and the spectral τ approximation using the Coulomb gauge in a density-stratified turbulence. The turbulent magnetic helicity fluxes contain non-gradient contribution that is proportional to the effective pumping velocity multiplied by the small-scale magnetic helicity. There is the gradient contribution to the turbulent magnetic helicity flux describing the turbulent magnetic diffusion of the small-scale magnetic helicity. The turbulent magnetic helicity flux includes also the source term proportional to the kinetic α effect or its radial gradient. Finally, there is a contribution to the turbulent magnetic helicity flux due to the solar differential rotation.

The convective zone of the Sun and solar-like stars as well as galactic discs are the source for production of turbulent magnetic helicity fluxes. The turbulent magnetic helicity flux due to the kinetic α effect and its radial derivative in combination with the turbulent magnetic diffusion of the small-scale magnetic helicity are dominant in the solar convective zone. The turbulent magnetic helicity fluxes result in evacuation of small-scale magnetic helicity from the regions of generation of the solar magnetic field, which allows to avoid the catastrophic quenching of the α effect. The small-scale magnetic helicity is not accumulated inside the solar convective zone due to turbulent magnetic diffusion flux.

The magnetic helicity fluxes are measured in the solar surface. Most of the measurements of the magnetic helicity fluxes are performed in active regions. The contributions to the measured magnetic helicity flux are from both, the solar surface and solar interiors.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

There are no new data associated with this article.

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APPENDIX A: DERIVATION OF TURBULENT FLUX OF MAGNETIC HELICITY

In this Section we derive turbulent flux of the magnetic helicity. We consider developed turbulence with large fluid and magnetic Reynolds numbers, so that the Strouhal number (the ratio of turbulent time τ to turn-over time ℓ_0/u_0) is of the order of unity, and the turbulent correlation time is scale-dependent, like in Kolmogorov type turbulence. In this case, we perform the Fourier transformation only in $\boldsymbol{k}$ space but not in ω space, as is usually done in studies of turbulent transport in a fully developed Kolmogorov-type turbulence. We take into account the nonlinear terms in equations for velocity and magnetic fluctuations and apply the τ approach.

The τ approach is a universal tool in turbulent transport for strongly nonlinear systems that allows us to obtain closed results and compare them with the results of laboratory experiments, observations, and numerical simulations. The τ approximation reproduces many well-known phenomena found by other methods in turbulent transport of particles and magnetic fields, in turbulent convection and stably stratified turbulent flows for large fluid and magnetic Reynolds and Péclet numbers.

To derive equations for the turbulent fluxes of the magnetic helicity, we need expressions in a Fourier space for the cross-helicity tensor $g_{ij}(\boldsymbol{k}) = \langle u_i(t,\boldsymbol{k})\, b_j(t,-\boldsymbol{k})\rangle$ and the tensor $h_{ij}(\boldsymbol{k}) = \langle b_i(t,\boldsymbol{k})\, b_j(t,-\boldsymbol{k})\rangle$ for magnetic fluctuations. Indeed, as follows from Eq. (8), the turbulent fluxes of the magnetic helicity depend only on the second moments g_{ij} and h_{ij} (except for the last two terms, $\eta\, \langle \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})\rangle$ and $\boldsymbol{F}^{\rm (III)}$ which are considered separately). Using induction equation (3) for magnetic fluctuations $\boldsymbol{b}$ and the Navier-Stokes equation for velocity fluctuations $\boldsymbol{u}$ written in a Fourier space, we derive equations for the cross-helicity tensor $g_{ij}(\boldsymbol{k})$ and the tensor $h_{ij}(\boldsymbol{k})$ for magnetic fluctuations as

$$
\frac{\partial g_{ij}(\boldsymbol{k})}{\partial t} = - \left[{\rm i}\, \boldsymbol{k}{\bf\cdot}\overline{\boldsymbol{B}} - \frac{1}{2}\overline{\boldsymbol{B}}{\bf\cdot}\boldsymbol{\nabla}\right] \left[f_{ij}(\boldsymbol{k}) - h_{ij}(\boldsymbol{k})\right] + \hat{\mathcal{M}}^{(b)} g_{ij}^{(III)}(\boldsymbol{k}), \tag{A1}
$$

$$
\frac{\partial h_{ij}(\boldsymbol{k})}{\partial t} = {\rm i}\left(\boldsymbol{k}{\bf\cdot}\overline{\boldsymbol{B}}\right)\left[g_{ij}(\boldsymbol{k}) - g_{ji}(-\boldsymbol{k})\right] + \frac{1}{2}\left(\overline{\boldsymbol{B}}{\bf\cdot}\boldsymbol{\nabla}\right)\left[g_{ij}(\boldsymbol{k}) + g_{ji}(-\boldsymbol{k})\right] + \hat{\mathcal{M}}^{(b)} h_{ij}^{(III)}(\boldsymbol{k}), \tag{A2}
$$

where in Eqs. (A1)–(A2) we neglect terms proportional to spatial derivatives of the mean magnetic field [i.e., terms $\propto$ O$(\nabla_i \overline{B}_j)$]. Here $f_{ij}(\boldsymbol{k}) = \langle u_i(t,\boldsymbol{k})\, u_j(t,-\boldsymbol{k})\rangle$, and $\hat{\mathcal{M}}^{(b)} g_{ij}^{(III)}$ and $\hat{\mathcal{M}}^{(b)} h_{ij}^{(III)}$ are the third-order moment terms appearing due to the nonlinear terms:

$$
\hat{\mathcal{M}}^{(b)} g_{ij}^{(III)}(\boldsymbol{k}) = - \left\langle u_i(t,\boldsymbol{k})\, T_j^{(b)}(t,-\boldsymbol{k})\right\rangle + \left\langle \frac{\partial u_i(t,\boldsymbol{k})}{\partial t}\, b_j(t,-\boldsymbol{k})\right\rangle, \tag{A3}
$$

$$
\hat{\mathcal{M}}^{(b)} h_{ij}^{(III)}(\boldsymbol{k}) = - \left\langle b_i(t,\boldsymbol{k})\, T_j^{(b)}(t,-\boldsymbol{k})\right\rangle - \left\langle T_i^{(b)}(t,\boldsymbol{k})\, b_j(t,-\boldsymbol{k})\right\rangle, \tag{A4}
$$

where

$$
T_j^{(b)} = \left[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{b} - \langle\boldsymbol{u}\times\boldsymbol{b}\rangle)\right]_j . \tag{A5}
$$

Equations (A1)-(A2) for the second moment includes the first-order spatial differential operators applied to the third-order moments $\hat{\mathcal{M}}^{(b)} g_{ij}^{(III)}(\boldsymbol{k})$ and $\hat{\mathcal{M}}^{(b)} h_{ij}^{(III)}(\boldsymbol{k})$. A problem arises how to close the system, i.e., how to express the third-order moments through the lower moments, g_{ij} and h_{ij} denoted as $F^{(II)}$. We use the spectral τ approximation which postulates that the deviations of the third-order moments, denoted as $\hat{\mathcal{M}}F^{(III)}(\boldsymbol{k})$, from the contributions to these terms afforded by a background turbulence, $\hat{\mathcal{M}}F^{(III,0)}(\boldsymbol{k})$, can be expressed through the similar deviations of the second moments, $F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k})$ as

$$
\hat{\mathcal{M}}F^{(III)}(\boldsymbol{k}) - \hat{\mathcal{M}}F^{(III,0)}(\boldsymbol{k}) = - \frac{1}{\tau_r(k)}\left[F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k})\right], \tag{A6}
$$

where $\tau_r(k)$ is the scale-dependent relaxation time, which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large fluid and magnetic Reynolds numbers. The functions with the superscript (0) correspond to the background turbulence with a zero mean magnetic field. Validation of the τ approximation for different situations has been performed in various numerical simulations (Brandenburg et al. 2004, 2008, 2012; Brandenburg & Subramanian 2005b,c,a; Rädler et al. 2011; Rogachevskii et al. 2011, 2012, 2018; Haugen et al. 2012; Elperin et al. 2017). When the mean magnetic field is zero, the turbulent electromotive force vanishes, which implies that $g_{ij}^{(0)}(\boldsymbol{k}) = 0$. We also take into account magnetic fluctuations caused by a small-scale dynamo (the dynamo with a zero mean magnetic field). Consequently, Eq. (A6) reduces to $\hat{\mathcal{M}}^{(b)} g_{ij}^{(III)}(\boldsymbol{k}) = -g_{ij}(\boldsymbol{k})/\tau(k)$ and $\hat{\mathcal{M}}^{(b)} h_{ij}^{(III)}(\boldsymbol{k}) = -[h_{ij}(\boldsymbol{k}) - h_{ij}^{(0)}(\boldsymbol{k})]/\tau(k)$.

We assume that the characteristic time of variation of the second moments $g_{ij}(\boldsymbol{k})$ and $h_{ij}(\boldsymbol{k})$ are substantially larger than the correlation time $\tau(k)$ for all turbulence scales. Therefore, in a steady-state Eqs. (A1) and (A2) yield the following formulae for the cross-helicity tensor $g_{ij}(\boldsymbol{k}) = \langle u_i(\boldsymbol{k})\, b_j(-\boldsymbol{k})\rangle$, and the function $h_{ij}(\boldsymbol{k}) = \langle b_i(\boldsymbol{k})\, b_j(-\boldsymbol{k})\rangle$:

$$
g_{ij}(\boldsymbol{k}) = -\tau(k)\left\{\left[{\rm i}\left(\boldsymbol{k}{\bf\cdot}\overline{\boldsymbol{B}}\right) - \frac{1}{2}\left(\overline{\boldsymbol{B}}{\bf\cdot}\boldsymbol{\nabla}\right)\right]\left[f_{ij}(\boldsymbol{k}) - h_{ij}(\boldsymbol{k})\right] - \overline{B}_j\left({\rm i}\, k_n - \frac{1}{2}\nabla_n\right) f_{in}(\boldsymbol{k})\right\}, \tag{A7}
$$

$$
h_{ij}(\boldsymbol{k}) = h_{ij}^{(0)}(\boldsymbol{k}) + \tau^2(k)\left(\boldsymbol{k}{\bf\cdot}\overline{\boldsymbol{B}}\right)\left[2\left(\boldsymbol{k}{\bf\cdot}\overline{\boldsymbol{B}}\right) f_{ij}(\boldsymbol{k}) - k_n\left(\overline{B}_j\, f_{in}(\boldsymbol{k}) + \overline{B}_i\, f_{nj}(\boldsymbol{k})\right)\right]. \tag{A8}
$$

In Eqs. (A7)–(A8) we neglect small contributions proportional to spatial derivatives of the mean magnetic field. Since we consider a one way coupling (i.e., we do not consider the algebraic quenching of the turbulent fluxes of the magnetic helicity), the correlation functions f_{ij} and h_{ij} in the right-hand sides of Eqs. (A7)–(A8) should be replaced by $f_{ij}^{(0)}$ and $h_{ij}^{(0)}$, respectively.

We use the following model for the second moment, $f_{ij}^{(0)}(\boldsymbol{k},\boldsymbol{R}) = \langle u_i(\boldsymbol{k})\, u_j(-\boldsymbol{k})\rangle^{(0)}$ of velocity fluctuations in density stratified and helical turbulence in a Fourier space (Rädler et al. 2003):

$$
f_{ij}^{(0)} = \frac{E_u(k)}{8\pi k^2}\left\{\left[(\delta_{ij} - k_{ij}) + \frac{\rm i}{k^2}\left(\tilde\lambda_i k_j - \tilde\lambda_j k_i\right)\right]\langle \boldsymbol{u}^2\rangle - \frac{1}{k^2}\left[{\rm i}\varepsilon_{ijp}\, k_p + (\varepsilon_{jpm}\, k_{ip} + \varepsilon_{ipm}\, k_{jp})\tilde\lambda_m\right] H_{\rm u}\right\}, \tag{A9}
$$

where δ_{ij} is the Kronecker tensor, $k_{ij} = k_i\, k_j/k^2$ and $\tilde\lambda_m = \lambda_m - \nabla_m/2$. The energy spectrum function $E_u(k)$ of velocity fluctuations in the inertial range of turbulence is given by $E_u(k) = (q-1)\, k_0^{-1}\, (k/k_0)^{-q}$, where the exponent $q = 5/3$ corresponds to the Kolmogorov spectrum, $k_0 \leqslant k \leqslant k_\nu$, the wave number $k_0 = 1/\ell_0$, the length ℓ_0 is the maximum scale of random motions, the wave number $k_\nu = \ell_\nu^{-1}$, the length $\ell_\nu = \ell_0 {\rm Re}^{-3/4}$ is the Kolmogorov (viscous) scale. The expression for the turbulent correlation time is given by $\tau(k) = 2\,\tau_0\,(k/k_0)^{1-q}$, where $\tau_0 = \ell_0/u_0$ is the characteristic turbulent time. In Eq. (A9) we take into account inhomogeneity of the kinetic helicity.

The model for the second moment, $h_{ij}^{(0)}(\boldsymbol{k},\boldsymbol{R}) = \langle b_i(\boldsymbol{k})\, b_j(-\boldsymbol{k})\rangle^{(0)}$, of magnetic fluctuations in a Fourier space is analogous to equation (A9)

$$h_{ij}^{(0)} = \frac{1}{8\pi k^2} \biggl\{ E_b(k) \, (\delta_{ij} - k_{ij}) \, \langle \boldsymbol{b}^2 \rangle - \frac{1}{k^2} \biggl[{\rm i}\varepsilon_{ijp} \, k_p$$
$$- \frac{1}{2} \, (\varepsilon_{jpm} \, k_{ip} + \varepsilon_{ipm} \, k_{jp}) \nabla_m \biggr] \, H_c \, \delta(k - k_0) \biggr\}, \qquad \text{(A10)}$$

where $H_c = \langle \boldsymbol{b} \boldsymbol{\cdot} (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle$ is the current helicity, $E_b(k) = (q_m - 1) \, k_b^{-1} \, (k/k_b)^{-q_m}$ is the magnetic energy spectrum function in the range $k_b \leqslant k \leqslant k_\eta$, the wave number $k_b = 1/\ell_b$, the length ℓ_b is the maximum scale of magnetic fluctuations caused by the small-scale dynamo, and the exponent $q_m = 5/3$ corresponds to the Kolmogorov spectrum for the magnetic energy. In Eq. (A10) we take into account inhomogeneity of the current helicity. We also take into account that due to the realizability condition, the current helicity of the small-scale field is located at the integral turbulence scale (Kleeorin & Rogachevskii 1999).

For the integration over angles in $\boldsymbol{k}$-space we use the following integrals:

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\vartheta \, d\vartheta \, k_{ij} = \frac{4\pi}{3} \, \delta_{ij}, \qquad \text{(A11)}$$

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\vartheta \, d\vartheta \, k_{ijmn} = \frac{4\pi}{15} \, \Delta_{ijmn}, \qquad \text{(A12)}$$

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\vartheta \, d\vartheta \, k_{ijmnpq} = \frac{4\pi}{105} \, \Delta_{ijmnpq}, \qquad \text{(A13)}$$

where

$$\Delta_{ijmn} = \delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}, \qquad \text{(A14)}$$

$$\Delta_{ijmnpq} = \Delta_{mnpq} \, \delta_{ij} + \Delta_{jmnq} \, \delta_{ip} + \Delta_{imnq} \, \delta_{jp}$$
$$+ \Delta_{jmnp} \, \delta_{iq} + \Delta_{imnp} \, \delta_{jq} + \Delta_{ijmn} \, \delta_{pq} - \Delta_{ijpq} \, \delta_{mn}, \qquad \text{(A15)}$$

and $k_{ij} = k_i \, k_j / k^2$, $k_{ijmn} = k_i \, k_j \, k_m \, k_n / k^4$ and $k_{ijmnpq} = k_i \, k_j \, k_m \, k_n \, k_p \, k_q / k^6$. We also take into account that $\Delta_{ijmm} = 5 \delta_{ij}$ and $\Delta_{ijmnpp} = 7 \Delta_{ijmn}$.

For the integration over k we use the following integrals for large Reynolds numbers, Re$= u_0 \ell_0 / \nu \gg 1$:

$$\int_{k_0}^{k_\nu} \tau(k) \, E_u(k) \, dk = \tau_0, \qquad \text{(A16)}$$

$$\int_{k_0}^{k_\nu} \frac{\tau(k) \, E_u(k)}{k^2} \, dk = \frac{q-1}{q} \, \tau_0 \, \ell_0^2, \qquad \text{(A17)}$$

$$\int_{k_0}^{k_\nu} \frac{\tau^2(k) \, E_u(k)}{k^2} \, dk = \frac{4(q-1)}{3q-1} \, \tau_0^2 \, \ell_0^2, \qquad \text{(A18)}$$

$$\int_{k_0}^{k_\nu} \tau^2(k) \, E_u(k) \, dk = \frac{4}{3} \tau_0^2. \qquad \text{(A19)}$$

Using Eqs. (A7)–(B15), and integrating in $\boldsymbol{k}$ space, we determine various contributions (8) to the turbulent flux of the small-scale magnetic helicity, see also Appendix D. The details of the derivations of the effect of large-scale shear on turbulent fluxes of the magnetic helicity are discussed in Appendix C.

APPENDIX B: DERIVATION OF EQUATIONS FOR THE SECOND MOMENTS

In this Appendix we derive Eqs. (A1)–(A2) for the cross helicity tensor $g_{ij}(\boldsymbol{k}) = \langle u_i(t, \boldsymbol{k}) \, b_j(t, -\boldsymbol{k}) \rangle$ and the tensor $h_{ij}(\boldsymbol{k}) = \langle b_i(t, \boldsymbol{k}) \, b_j(t, -\boldsymbol{k}) \rangle$ for magnetic fluctuations. To this end, we perform several calculations that are similar to the following. We use the equation for magnetic fluctuations obtained by subtracting equation for the mean magnetic field from the equation for the total field:

$$\frac{\partial \boldsymbol{b}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle) - \eta \, \Delta \boldsymbol{b} = (\overline{\boldsymbol{B}} \boldsymbol{\cdot} \boldsymbol{\nabla}) \boldsymbol{u} - (\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla}) \overline{\boldsymbol{B}}. \qquad \text{(B1)}$$

The source term, $(\overline{\boldsymbol{B}} \boldsymbol{\cdot} \boldsymbol{\nabla}) \boldsymbol{u}$, in the right hand side of Eq. (B1) in a Fourier space reads:

$$\left[(\overline{\boldsymbol{B}} \boldsymbol{\cdot} \boldsymbol{\nabla}) \, u_j \right]_{\boldsymbol{k}} = {\rm i} \, k_p \int \overline{B}_p(\boldsymbol{Q}) \, u_j(\boldsymbol{k} - \boldsymbol{Q}) \, d\boldsymbol{Q}, \qquad \text{(B2)}$$

so that the induction equation for $b_j(\boldsymbol{k}_2)$ in $\boldsymbol{k}$ space is given by:

$$\frac{\partial b_j(\boldsymbol{k}_2)}{\partial t} = {\rm i} \, k_p^{(2)} \int \overline{B}_p(\boldsymbol{Q}) \, u_j(\boldsymbol{k}_2 - \boldsymbol{Q}) \, d\boldsymbol{Q}$$
$$- u_n(\boldsymbol{k}_2) \, \nabla_n \overline{B}_j + N_j^{(b)}(\boldsymbol{k}_2), \qquad \text{(B3)}$$

where $\boldsymbol{k}^{(2)} \equiv \boldsymbol{k}_2 = -\boldsymbol{k} + \boldsymbol{K}/2$. We use the identity:

$$\frac{\partial}{\partial t} \langle u_i(\boldsymbol{k}_1, t) \, b_j(\boldsymbol{k}_2, t) \rangle = \left\langle \frac{\partial u_i(\boldsymbol{k}_1, t)}{\partial t} \, b_j(\boldsymbol{k}_2, t) \right\rangle$$
$$+ \left\langle u_i(\boldsymbol{k}_1, t) \, \frac{\partial b_j(\boldsymbol{k}_2, t)}{\partial t} \right\rangle. \qquad \text{(B4)}$$

First we derive equation for the second term in the right hand side of Eq. (B4). To this end, we multiply Eq. (B3) by $u_i(\boldsymbol{k}_1)$ and averaging over ensemble of turbulent velocity field, where $\boldsymbol{k}_1 = \boldsymbol{k} + \boldsymbol{K}/2$. This yields:

$$\left\langle u_i(\boldsymbol{k}_1) \, \frac{\partial b_j(\boldsymbol{k}_2)}{\partial t} \right\rangle = {\rm i} \, (-k_p + K_p/2) \int d\boldsymbol{Q} \, \overline{B}_p(\boldsymbol{Q})$$
$$\times \langle u_i(\boldsymbol{k}_1) \, u_j(\boldsymbol{k}_2 - \boldsymbol{Q}) \rangle - \langle u_i(\boldsymbol{k}_1) \, u_n(\boldsymbol{k}_2) \rangle \, \nabla_n \overline{B}_j$$
$$+ \left\langle u_i(\boldsymbol{k}_1) \, N_j^{(b)}(\boldsymbol{k}_2) \right\rangle, \qquad \text{(B5)}$$

where for brevity of notations we omit the argument t in the velocity and magnetic fields. Next, we perform in Eq. (B5) the Fourier transformation in the large-scale variable $\boldsymbol{K}$, i.e., we use the transformation

$$F(\boldsymbol{R}) = \int F(\boldsymbol{K}) \exp({\rm i} \, \boldsymbol{K} \boldsymbol{\cdot} \boldsymbol{R}) \, d\boldsymbol{K}.$$

The first term $S_{ij}(\boldsymbol{k}, \boldsymbol{R})$ in the right hand side of the obtained equation [which originates from the first term in the right hand side of Eq. (B3)], is given by:

$$S_{ij}(\boldsymbol{k}, \boldsymbol{R}) = {\rm i} \int \int \overline{B}_p(\boldsymbol{Q}) \, (-k_p + K_p/2) \, \exp({\rm i} \, \boldsymbol{K} \boldsymbol{\cdot} \boldsymbol{R})$$
$$\times \langle u_i(\boldsymbol{k} + \boldsymbol{K}/2) u_j(-\boldsymbol{k} + \boldsymbol{K}/2 - \boldsymbol{Q}) \rangle \, d\boldsymbol{K} \, d\boldsymbol{Q}. \qquad \text{(B6)}$$

Next, we introduce new variables:

$$\tilde{\boldsymbol{k}} = (\tilde{\boldsymbol{k}}_1 - \tilde{\boldsymbol{k}}_2)/2 = \boldsymbol{k} + \boldsymbol{Q}/2,$$
$$\tilde{\boldsymbol{K}} = \tilde{\boldsymbol{k}}_1 + \tilde{\boldsymbol{k}}_2 = \boldsymbol{K} - \boldsymbol{Q}, \qquad \text{(B7)}$$

where

$$\tilde{\boldsymbol{k}}_1 = \boldsymbol{k} + \boldsymbol{K}/2, \quad \tilde{\boldsymbol{k}}_2 = -\boldsymbol{k} + \boldsymbol{K}/2 - \boldsymbol{Q}. \qquad \text{(B8)}$$

Therefore, Eq. (B6) in the new variables reads

$$S_{ij}(\boldsymbol{k},\boldsymbol{R}) = \mathrm{i} \int\int f_{ij}\,(\boldsymbol{k}+\boldsymbol{Q}/2, \boldsymbol{K}-\boldsymbol{Q})\, \overline{B}_p(\boldsymbol{Q}) \times(-k_p + K_p/2)\, \exp(\mathrm{i}\,\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R})\, d\boldsymbol{K}\, d\boldsymbol{Q}. \tag{B9}$$

Since $|\boldsymbol{Q}| \ll |\boldsymbol{k}|$, we use the Taylor expansion

$$f_{ij}(\boldsymbol{k}+\boldsymbol{Q}/2, \boldsymbol{K}-\boldsymbol{Q}) \simeq f_{ij}(\boldsymbol{k}, \boldsymbol{K}-\boldsymbol{Q}) + \frac{1}{2}\frac{\partial f_{ij}(\boldsymbol{k},\boldsymbol{K}-\boldsymbol{Q})}{\partial k_s} Q_s + O(\boldsymbol{Q}^2), \tag{B10}$$

and the following identity:

$$\nabla_p[f_{ij}(\boldsymbol{k},\boldsymbol{R})\overline{B}_p(\boldsymbol{R})] = \mathrm{i}\int d\boldsymbol{K}\, K_p[f_{ij}(\boldsymbol{k},\boldsymbol{R})\overline{B}_p(\boldsymbol{R})]_{\boldsymbol{K}} \times \exp(\mathrm{i}\,\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R}), \tag{B11}$$

where

$$[f_{ij}(\boldsymbol{k},\boldsymbol{R})\overline{B}_p(\boldsymbol{R})]_{\boldsymbol{K}} = \int f_{ij}(\boldsymbol{k},\boldsymbol{K}-\boldsymbol{Q})\overline{B}_p(\boldsymbol{Q})\, d\boldsymbol{Q}. \tag{B12}$$

Therefore, Eqs. (B9)–(B11) yield

$$S_{ij}(\boldsymbol{k},\boldsymbol{R}) \simeq \left[-\mathrm{i}\,(\boldsymbol{k}\cdot\overline{\boldsymbol{B}}) + \frac{1}{2}(\overline{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\right] f_{ij}(\boldsymbol{k},\boldsymbol{R}) - \frac{1}{2}k_p \frac{\partial f_{ij}(\boldsymbol{k})}{\partial k_s}\nabla_s \overline{B}_p. \tag{B13}$$

We take into account that the terms in $g_{ij}(\boldsymbol{k},\boldsymbol{R})$ with symmetric tensors with respect to the indexes "i" and "j" do not contribute to the turbulent electromotive force because $\mathcal{E}_m = \varepsilon_{mij} \int g_{ij}(\boldsymbol{k},\boldsymbol{R})\, d\boldsymbol{k}$. In $g_{ij}(\boldsymbol{k},\boldsymbol{R})$ we also neglect the second and higher derivatives over $\boldsymbol{R}$. This procedure yields Eq. (A1). Similar calculations are performed to derive Eq. (A2).

To determine various contributions to the turbulent flux of small-scale magnetic helicity, we use the following identities:

$$\left(\Delta^{-1}\right)_{\boldsymbol{k}_1} = -k^{-2}\left[1 + \frac{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{\nabla})}{k^2}\right], \tag{B14}$$

$$\left(\Delta^{-1}\right)_{\boldsymbol{k}_2} = -k^{-2}\left[1 - \frac{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{\nabla})}{k^2}\right]. \tag{B15}$$

APPENDIX C: EFFECT OF LARGE-SCALE SHEAR

In this Appendix we determine the effect of large-scale shear on turbulent fluxes of the magnetic helicity. The cross-helicity tensor $g_{ij}^{(S)}(\boldsymbol{k}) = \langle v_i(\boldsymbol{k})\, b_j(-\boldsymbol{k})\rangle$ in turbulence with large-scale shear is given by (Rogachevskii & Kleeorin 2004):

$$g_{ij}^{(S)}(\boldsymbol{k}) = -\mathrm{i}\,\tau\,(\boldsymbol{k}\cdot\overline{\boldsymbol{B}})\left[f_{ij}^{(S)}(\boldsymbol{k}) - \frac{h_{ij}^{(S)}(\boldsymbol{k})}{4\pi\overline{\rho}} + \tau\, J_{ijmn}(\overline{\boldsymbol{U}})\left(f_{mn}^{(0)}(\boldsymbol{k}) - \frac{h_{mn}^{(0)}(\boldsymbol{k})}{4\pi\overline{\rho}}\right)\right], \tag{C1}$$

where the effect of large-scale shear on the tensors $f_{ij}^{(S)}(\boldsymbol{k}) = \langle v_i(\boldsymbol{k})\, v_j(-\boldsymbol{k})\rangle$ and $h_{ij}^{(S)}(\boldsymbol{k}) = \langle b_i(\boldsymbol{k})\, b_j(-\boldsymbol{k})\rangle$ is determined by

$$f_{ij}^{(S)}(\boldsymbol{k}) = \tau\, I_{ijmn}(\overline{\boldsymbol{U}})\, f_{mn}^{(0)}(\boldsymbol{k}), \tag{C2}$$

$$h_{ij}^{(S)}(\boldsymbol{k}) = \tau\, E_{ijmn}(\overline{\boldsymbol{U}})\, h_{mn}^{(0)}(\boldsymbol{k}), \tag{C3}$$

and the tensors $I_{ijmn}(\overline{\boldsymbol{U}})$, $E_{ijmn}(\overline{\boldsymbol{U}})$ and $J_{ijmn}(\overline{\boldsymbol{U}})$ are given by

$$I_{ijmn}(\overline{\boldsymbol{U}}) = \Big\{2k_{iq}\delta_{mp}\delta_{jn} + 2k_{jq}\delta_{im}\delta_{pn} - \delta_{im}\delta_{jq}\delta_{np} - \delta_{iq}\delta_{jn}\delta_{mp} + 4k_{pq}\delta_{im}\delta_{jn} + \delta_{im}\delta_{jn}k_q\frac{\partial}{\partial k_p} - \frac{\mathrm{i}\,\lambda_r}{2k^2}\Big[\Big(k_i\delta_{jn}\delta_{pm} - k_j\delta_{im}\delta_{pn}\Big)\Big(2k_{rq} - \delta_{rq}\Big) + k_q\Big(\delta_{ip}\delta_{jn}\delta_{rm} - \delta_{im}\delta_{jp}\delta_{rn}\Big) - 2k_{pq}\Big(k_i\delta_{jn}\delta_{rm} - k_j\delta_{im}\delta_{rn}\Big)\Big]\Big\}\nabla_p\overline{U}_q, \tag{C4}$$

$$E_{ijmn}(\overline{\boldsymbol{U}}) = \Big[\delta_{im}\delta_{jq}\delta_{pn} + \delta_{iq}\delta_{jn}\delta_{pm} + \delta_{im}\delta_{jn}k_q\frac{\partial}{\partial k_p}\Big]\nabla_p\overline{U}_q\ , \tag{C5}$$

$$J_{ijmn}(\overline{\boldsymbol{U}}) = \Big\{2k_{iq}\delta_{jn}\delta_{pm} - \delta_{iq}\delta_{jn}\delta_{pm} + \delta_{im}\delta_{jq}\delta_{pl} + 2k_{pq}\delta_{im}\delta_{jn} + \delta_{im}\delta_{jn}k_q\frac{\partial}{\partial k_p} - \frac{\mathrm{i}\,\lambda_r}{2k^2}\Big[k_i\delta_{jn}\delta_{pm} \times\Big(2k_{rq} - \delta_{rq}\Big) + \delta_{jn}\delta_{rm}\Big(k_q\,\delta_{ip} - 2k_i\,k_{pq}\Big)\Big]\Big\}\nabla_p\overline{U}_q. \tag{C6}$$

Using Eqs. (A9)–(B15) and Eqs. (C1)–(C6), and integrating in $\boldsymbol{k}$ space, we determine various contributions (8) to the turbulent flux of the small-scale magnetic helicity caused by the differential rotation, see Appendix D.

APPENDIX D: GENERAL FORM OF TURBULENT TRANSPORT COEFFICIENTS

Applying the method described in Appendixes A–C, we have determined various contributions to the turbulent flux of the small-scale magnetic helicity. In particular, the general form of turbulent flux of the small-scale magnetic helicity is given by

$$F_i^{(\mathrm{m})} = V_i^{(\mathrm{H})}\, H_{\mathrm{m}} - D_{ij}^{(\mathrm{H})}\,\nabla_j H_{\mathrm{m}} + N_i^{(\alpha)}\,\alpha_{\mathrm{K}} + M_{ij}^{(\alpha)}\,\nabla_j\alpha_{\mathrm{K}} + F_i^{(\mathrm{S0})}, \tag{D1}$$

where the turbulent transport coefficients are given below. The turbulent pumping velocity $\boldsymbol{V}^{(\mathrm{H})}$ of the small-scale magnetic helicity is

$$\boldsymbol{V}^{(\mathrm{H})} = -\frac{1}{15}\tau_0\,\overline{V}_{\mathrm{A}}^2\Big\{\boldsymbol{\lambda} + 7\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{\lambda}) + \frac{1}{7}\tau_0\Big[28\,(\overline{\boldsymbol{W}}\times\boldsymbol{\lambda}) + \frac{139}{2}(\boldsymbol{\beta}\cdot\boldsymbol{\lambda})\,(\overline{\boldsymbol{W}}\times\boldsymbol{\beta}) - 2\boldsymbol{Q}^{(\lambda)} + \boldsymbol{\beta}\Big(17\,\overline{\boldsymbol{W}}\cdot(\boldsymbol{\beta}\times\boldsymbol{\lambda}) + 58\,\boldsymbol{\lambda}\cdot\boldsymbol{Q}^{(\beta)}\Big) - 31\,\boldsymbol{Q}^{(\beta)}\,(\boldsymbol{\beta}\cdot\boldsymbol{\lambda}) - 3\,\boldsymbol{\lambda}(\boldsymbol{\beta}\cdot\boldsymbol{Q}^{(\beta)}) - 7\,(\boldsymbol{\beta}\times\boldsymbol{\lambda})\,(\boldsymbol{\beta}\cdot\overline{\boldsymbol{W}})\Big]\Big\}. \tag{D2}$$

Here $\boldsymbol{\beta} = \overline{\boldsymbol{B}}/\overline{B}$ is the unit vector along the mean magnetic field, $\overline{V}_{\rm A} = \overline{B}/(4\pi\overline{\rho})^{1/2}$ is the mean Alfvén speed, $\overline{\boldsymbol{W}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$ is the mean vorticity, the vectors $\boldsymbol{Q}^{(\beta)}$ and $\boldsymbol{Q}^{(\lambda)}$ are defined as $Q_i^{(\beta)} = \beta_m\,(\partial\overline{U})_{mi}$ and $Q_i^{(\lambda)} = \lambda_m\,(\partial\overline{U})_{mi}$, and the gradient of the mean velocity $\nabla_i\overline{U}_j$ is decomposed into symmetric, $(\partial\overline{U})_{ij} = (\nabla_i\overline{U}_j + \nabla_j\overline{U}_i)/2$, and antisymmetric, $\varepsilon_{ijp}\,\overline{W}_p/2$ parts, i.e., $\nabla_i\overline{U}_j = (\partial\overline{U})_{ij} + \varepsilon_{ijp}\,\overline{W}_p/2$.

The total diffusion tensor $D_{ij}^{(\rm H)}$ that describes the microscopic and turbulent magnetic diffusion of the small-scale magnetic helicity, reads:

$$
\begin{aligned}
D_{ij}^{(\rm H)} &= D_T^{(\rm H)}\,\delta_{ij} + \frac{1}{30}\tau_0\,\overline{V}_{\rm A}^2 \biggl\{5\delta_{ij} - 4\beta_i\,\beta_j + \tau_0\biggl[8\varepsilon_{ijp} \\
&\times(\overline{\boldsymbol{W}}\cdot\boldsymbol{\beta})\,\beta_p + 8\,\beta_i\,(\boldsymbol{\beta}\times\overline{\boldsymbol{W}})_j + 14\,\beta_j\,(\boldsymbol{\beta}\times\overline{\boldsymbol{W}})_i \\
&+4\,\varepsilon_{iqm}\,\varepsilon_{jpn}\,\beta_m\,\beta_n\,(\partial\overline{U})_{pq} + \frac{1}{7}\biggl(8(q+1)\,(\partial\overline{U})_{ij} \\
&+2(41+34q)\,\beta_i\,Q_j^{(\beta)} + 2(1-6q)\,\beta_j\,Q_i^{(\beta)} + (1+8q)\delta_{ij} \\
&\times(\boldsymbol{\beta}\cdot\boldsymbol{Q}^{(\beta)})\biggr)\biggr]\biggr\} + \frac{\tau_0}{2}\left[\eta_T + \frac{8}{15}\tau_0\,\overline{V}_{\rm A}^2\right]\varepsilon_{ijp}\,\overline{W}_p. \quad \text{(D3)}
\end{aligned}
$$

In derivation of Eqs. (D2)–(D3), we take into account that $H_{\rm c} = H_{\rm m}/\ell_0^2$, and we neglect small terms $\sim {\rm O}[\ell_0^2/L_{\rm m}^2]$ with $L_{\rm m}$ being characteristic scale of spatial variations of $H_{\rm m}$. The turbulent magnetic helicity flux also includes the source term $\boldsymbol{N}^{(\alpha)}\,\alpha_{\rm K}$ caused by the kinetic α effect with $\boldsymbol{N}^{(\alpha)}$ being

$$
\begin{aligned}
\boldsymbol{N}^{(\alpha)} &= -\frac{1}{10}\,\ell_0^2\,\overline{B}^2\biggl\{\boldsymbol{\lambda} + \frac{7q-2}{q}\,(\boldsymbol{\beta}\cdot\boldsymbol{\lambda})\,\boldsymbol{\beta} + \frac{(q-1)\,\tau_0}{(3q-1)} \\
&\times\biggl[10\,(\boldsymbol{\beta}\times\overline{\boldsymbol{W}})\,(\boldsymbol{\beta}\cdot\boldsymbol{\lambda}) - 37(\overline{\boldsymbol{W}}\cdot\boldsymbol{\beta})\,(\boldsymbol{\beta}\times\boldsymbol{\lambda}) - 4\boldsymbol{Q}^{(\lambda)} \\
&-4\,(\boldsymbol{\beta}\times\boldsymbol{Q}^{(\beta,\lambda)}) + \frac{2}{7}\biggl(19\,\boldsymbol{\beta}\,[(\boldsymbol{\beta}\times\overline{\boldsymbol{W}})\cdot\boldsymbol{\lambda}] - 4\,\boldsymbol{Q}^{(\beta)} \\
&\times(\boldsymbol{\beta}\cdot\boldsymbol{\lambda}) - 24\,\boldsymbol{\beta}\,(\boldsymbol{\lambda}\cdot\boldsymbol{Q}^{(\beta)}) + 4\,\boldsymbol{\lambda}\,(\boldsymbol{\beta}\cdot\boldsymbol{Q}^{(\beta)})\biggr)\biggr]\biggr\}, \quad \text{(D4)}
\end{aligned}
$$

where $Q_i^{(\beta,\lambda)} = (\boldsymbol{\beta}\times\boldsymbol{\lambda})_m\,(\partial\overline{U})_{mi}$. The contribution to the turbulent magnetic helicity flux, $\propto -\,\ell_0^2\,\overline{B}^2\,\boldsymbol{\lambda}\,\alpha_{\rm K}$ [see the first term in equation (D4)], caused by the kinetic α effect, has been suggested by Kleeorin et al. (2000, 2002, 2003a).

The turbulent magnetic helicity flux contains also the source term $M_{ij}^{(\alpha)}\,\nabla_j\alpha_{\rm K}$ caused by the gradient $\nabla_j\alpha_{\rm K}$ of the kinetic α effect with $M_{ij}^{(\alpha)}$ being

$$
\begin{aligned}
M_{ij}^{(\alpha)} &= \frac{1}{20q}\,\ell_0^2\,\overline{B}^2\biggl\{(2q-1)\,\delta_{ij} + (20q-23)\,\beta_i\,\beta_j \\
&+\frac{16\,q\,(q-1)\tau_0}{3q-1}\left[\beta_i\,(\boldsymbol{\beta}\times\overline{\boldsymbol{W}})_j + (\overline{\boldsymbol{W}}\cdot\boldsymbol{\beta})\,\varepsilon_{ijp}\,\beta_p\right]\biggr\}.
\end{aligned}
\quad \text{(D5)}
$$

The additional contribution $\boldsymbol{F}^{(\rm S0)}$ to the turbulent magnetic helicity flux caused by the large-scale shear (differential rotation) is given by

$$
\begin{aligned}
\boldsymbol{F}^{(\rm S0)} &= -\frac{q-1}{3(q+1)}\,\ell_b^2\,\langle\boldsymbol{b}^2\rangle\,\overline{\boldsymbol{W}} + \frac{2}{45}\,\ell_0^2\,\overline{B}^2\,\biggl[11\epsilon\,\overline{\boldsymbol{W}} \\
&+(3\epsilon-10)\,(\boldsymbol{\beta}\cdot\overline{\boldsymbol{W}})\,\boldsymbol{\beta} + (\boldsymbol{\beta}\times\boldsymbol{Q}^{(\beta)})[8q+35 \\
&+\epsilon(8q-20)]\biggr]. \quad \text{(D6)}
\end{aligned}
$$

Here $\epsilon = \ell_b^2\,\langle\boldsymbol{b}^2\rangle/(\ell_0^2\,4\pi\overline{\rho}\,\langle\boldsymbol{u}^2\rangle)$, and ℓ_b is the energy containing scale of magnetic fluctuations with a zero mean magnetic field. The contribution to the turbulent magnetic helicity flux, $\propto\,\ell_0^2\,\overline{B}^2\,(\boldsymbol{\beta}\times\boldsymbol{Q}^{(\beta)})$ [see the last term in equation (D6)], caused by the large-scale shear, has been derived by Brandenburg & Subramanian (2005a), using a general expression originally suggested by Vishniac & Cho (2001).

To derive equations for the turbulent magnetic helicity flux due to the differential rotation in spherical coordinates, we use the identities given below. The large-scale shear velocity $\overline{\boldsymbol{U}} = \boldsymbol{\delta\Omega}\times\boldsymbol{r}$ is caused by the differential (non-uniform) rotation, that is in spherical coordinates (r,ϑ,φ) reads

$$
\boldsymbol{\delta\Omega} = \delta\Omega(r,\vartheta)\,(\cos\vartheta, -\sin\vartheta, 0), \quad \text{(D7)}
$$

and the stress tensor $(\partial\overline{U})_{ij}$ reads

$$
(\partial\overline{U})_{ij} = \frac{r_n}{2}\,(\varepsilon_{imn}\nabla_j + \varepsilon_{jmn}\nabla_i)\,\delta\Omega_m. \quad \text{(D8)}
$$

The vectors $\boldsymbol{Q}^{(\beta)}$ and $\boldsymbol{Q}^{(\lambda)}$ defined as $Q_i^{(\beta)} = \beta_m\,(\partial\overline{U})_{mi}$ and $Q_i^{(\lambda)} = \lambda_m\,(\partial\overline{U})_{mi}$, are given by

$$
\boldsymbol{Q}^{(\beta)} \;=\; (\boldsymbol{r}\times\boldsymbol{\beta})_m\,(\boldsymbol{\nabla}\delta\Omega_m) - \boldsymbol{r}\times(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})\boldsymbol{\delta\Omega}, \quad \text{(D9)}
$$

$$
\boldsymbol{Q}^{(\lambda)} \;=\; -\boldsymbol{r}\times(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})\boldsymbol{\delta\Omega}, \quad \text{(D10)}
$$

where $\boldsymbol{\lambda} = \lambda\,\boldsymbol{e}_r$ and $\boldsymbol{\beta} = \overline{\boldsymbol{B}}/\overline{B} = (\beta_r, \beta_\vartheta, \beta_\varphi)$. We also use the identity

$$
\begin{aligned}
\varepsilon_{iqm}\,\varepsilon_{jpn}\,\beta_m\,\beta_n\,(\partial\overline{U})_{pq} &= \frac{1}{2}\,(\boldsymbol{r}\cdot\boldsymbol{\beta})\,\Bigl[(\boldsymbol{\beta}\times\boldsymbol{\nabla})_i\,\delta\Omega_j \\
&+(\boldsymbol{\beta}\times\boldsymbol{\nabla})_j\,\delta\Omega_i\Bigr] - \frac{1}{2}\,\beta_m\,\Bigl[r_i\,(\boldsymbol{\beta}\times\boldsymbol{\nabla})_j \\
&+r_j\,(\boldsymbol{\beta}\times\boldsymbol{\nabla})_i\Bigr]\,\delta\Omega_m. \quad \text{(D11)}
\end{aligned}
$$

We have taken into account that $\left(\boldsymbol{\beta}\times\boldsymbol{Q}^{(\beta)}\right)_r = {\rm O}(\boldsymbol{\nabla}\delta\Omega)$, i.e it does not contain contributions $\propto\delta\Omega$, but it includes their spatial derivatives, $\boldsymbol{\nabla}\delta\Omega$. Using Eqs. (D1)–(D11), we determine various contributions to the turbulent flux of the small-scale magnetic helicity in spherical coordinates, see Eqs. (11)–(18).

APPENDIX E: TURBULENT TRANSPORT COEFFICIENTS IN THE CARTESIAN COORDINATES

For better understanding of the physics related to various contributions to the turbulent flux of the small-scale magnetic helicity [see Eqs. (D1)–(D11)], we consider a small-scale turbulence with large-scale linear velocity shear $\overline{\boldsymbol{U}} = (0, Sx, 0)$ in the Cartesian coordinates. In this case, the large-scale vorticity is $\overline{\boldsymbol{W}} = (0,0,S)$, the stress tensor $(\partial\overline{U})_{ij} = (S/2)\,(e_i^x\,e_j^y + e_j^x\,e_i^y)$, the vector $\boldsymbol{\lambda}$ that describes the non-uniform mean fluid density, is $\boldsymbol{\lambda} = \lambda\,(\sin\vartheta, 0, \cos\vartheta)$, the unit vector along the large-scale magnetic is $\boldsymbol{\beta} = (\cos\tilde\beta, \sin\tilde\beta, 0)$, the vector $Q_i^{(\beta)} = \beta_m\,(\partial\overline{U})_{mi} = (S/2)\,(\sin\tilde\beta, \cos\tilde\beta, 0)$ and the vector $Q_i^{(\lambda)} = \lambda_m\,(\partial\overline{U})_{mi} = (\lambda\,S/2)\,\sin\vartheta\,e_i^y$. We also take into account that

$$
\boldsymbol{\beta}\times\boldsymbol{\lambda} = \lambda\,(\cos\vartheta\,\sin\tilde\beta, -\cos\vartheta\,\cos\tilde\beta, -\sin\vartheta\,\sin\tilde\beta), \quad \text{(E1)}
$$

$$
(\boldsymbol{\beta}\times\boldsymbol{Q}^{(\beta)})_i = (S/2)\,\cos(2\tilde\beta)\,e_i^z, \quad \text{(E2)}
$$

$$
(\boldsymbol{\beta}\times\boldsymbol{Q}^{(\lambda)})_i = (S\,\lambda/2)\,\sin\vartheta\,\cos\tilde\beta\,e_i^z, \quad \text{(E3)}
$$

$$\boldsymbol{\beta} \times \boldsymbol{W} = S\,(\sin\tilde{\beta}, -\cos\tilde{\beta}, 0), \tag{E4}$$

$$(\overline{\boldsymbol{W}} \times \boldsymbol{\lambda})_i = S\,\lambda\,\sin\vartheta\, e_i^y. \tag{E5}$$

First, we determine various contributions to the turbulent flux of the magnetic helicity inside the turbulent region where the toroidal mean magnetic field is much larger than the poloidal mean magnetic field, i.e., $\boldsymbol{\beta} = (0,1,0)$. In this case, the turbulent pumping velocity $\boldsymbol{V}^{(\mathrm{H})}$ of the small-scale magnetic helicity is

$$\boldsymbol{V}^{(\mathrm{H})} = -\frac{1}{15}\tau_0\,\overline{V}_{\rm A}^2\,\lambda\left[\left(1 + \frac{3}{14} S\,\tau_0\right)\boldsymbol{e}_\lambda + 5.6\,S\,\tau_0\,\boldsymbol{e}^y\right], \tag{E6}$$

where $\boldsymbol{e}_\lambda = \boldsymbol{\lambda}/\lambda$. The turbulent magnetic helicity flux has the source term $\boldsymbol{N}^{(\alpha)}\,\alpha_{\rm K}$ caused by the kinetic α effect with $\boldsymbol{N}^{(\alpha)}$ being

$$\boldsymbol{N}^{(\alpha)} = -\frac{1}{10}\,\ell_0^2\,\overline{B}^2\,\boldsymbol{\lambda}\left[1 - \frac{4(q-1)}{7(3q-1)}\,S\,\tau_0\right]. \tag{E7}$$

The total diffusion tensor $D_{ij}^{(\mathrm{H})}$ which describes the microscopic and turbulent magnetic diffusion of the small-scale magnetic helicity is given by:

$$D_{ij}^{(\mathrm{H})} = D_1\,\delta_{ij} - D_2 e_i^y\, e_j^y + D_3 e_i^x\, e_j^y - D_4 e_i^y\, e_j^x, \tag{E8}$$

where $D_2 = (2/15)\,\tau_0\,\overline{V}_{\rm A}^2$,

$$D_1 = D_T^{(\mathrm{H})} + \frac{1}{3}\eta + \frac{1}{6}\tau_0\,\overline{V}_{\rm A}^2\left[1 - \frac{1+8q}{70}\,S\,\tau_0\right], \tag{E9}$$

$$D_3 = \frac{1}{2}\,S\,\tau_0\left[\eta_T + \frac{159-6q}{105}\,\tau_0\,\overline{V}_{\rm A}^2\right], \tag{E10}$$

$$D_4 = \frac{1}{2}\,S\,\tau_0\left[\eta_T - \frac{34q+45}{105}\,\tau_0\,\overline{V}_{\rm A}^2\right]. \tag{E11}$$

Equation (E8) implies that $D_{xx}^{(\mathrm{H})} = D_{zz}^{(\mathrm{H})} = D_1$, $D_{yy}^{(\mathrm{H})} = D_1 - D_2$, $D_{xy}^{(\mathrm{H})} = D_3$, $D_{yx}^{(\mathrm{H})} = -D_4$, and other components of the total diffusion tensor $D_{ij}^{(\mathrm{H})}$ vanish. The turbulent magnetic helicity flux containing the source term $M_{ij}^{(\alpha)}\,\nabla_j \alpha_{\rm K}$ with $M_{ij}^{(\alpha)}$ being

$$M_{ij}^{(\alpha)} = \frac{1}{20q}\,\ell_0^2\,\overline{B}^2\left[(2q-1)\,\delta_{ij} + (20q-23)\,e_i^y\,e_j^y + \frac{16\,q\,(q-1)}{3q-1}\,S\,\tau_0\,e_i^y\,e_j^x\right]. \tag{E12}$$

The additional contribution $\boldsymbol{F}^{(\mathrm{S0})}$ to the turbulent magnetic helicity flux caused by the large-scale shear is given by

$$\boldsymbol{F}^{(\mathrm{S0})} = -\left[\frac{q-1}{3(q+1)} - \frac{22}{45}\,\frac{\overline{V}_{\rm A}^2}{\langle \boldsymbol{u}^2\rangle}\right]\ell_b^2\,\langle \boldsymbol{b}^2\rangle\,S\,\boldsymbol{e}^z. \tag{E13}$$

Now we determine various contributions to the turbulent flux of the magnetic helicity at the surface (the upper boundary of the turbulent region), where the toroidal mean magnetic field is much smaller than the poloidal mean magnetic field, i.e., $\boldsymbol{\beta} = (1,0,0)$. In this case, the turbulent pumping velocity $\boldsymbol{V}^{(\mathrm{H})}$ of the small-scale magnetic helicity is

$$\boldsymbol{V}^{(\mathrm{H})} = -\frac{1}{15}\tau_0\,\overline{V}_{\rm A}^2\,\lambda\left[\boldsymbol{e}_\lambda + 7\,\sin\vartheta\left(\boldsymbol{e}^x + \frac{81}{49}\,S\,\tau_0\,\boldsymbol{e}^y\right)\right]. \tag{E14}$$

The turbulent magnetic helicity flux has the source term $\boldsymbol{N}^{(\alpha)}\,\alpha_{\rm K}$ caused by the kinetic α effect with $\boldsymbol{N}^{(\alpha)}$ being

$$\boldsymbol{N}^{(\alpha)} = -\frac{1}{10}\,\ell_0^2\,\overline{B}^2\,\lambda\left[\boldsymbol{e}_\lambda + \frac{7q-2}{q}\,\sin\vartheta\,\boldsymbol{e}^x - \frac{2(q-1)}{3q-1}\,S\,\tau_0\left(\boldsymbol{e}^z + \frac{44}{7}\,\sin\vartheta\,\boldsymbol{e}^y\right)\right]. \tag{E15}$$

The total diffusion tensor $D_{ij}^{(\mathrm{H})}$ which describes the microscopic and turbulent magnetic diffusion of the small-scale magnetic helicity is given by:

$$D_{ij}^{(\mathrm{H})} = D_1\,\delta_{ij} - D_2 e_i^x\, e_j^x + D_3 e_i^x\, e_j^y - D_4 e_i^y\, e_j^x, \tag{E16}$$

where $D_2 = (2/15)\,\tau_0\,\overline{V}_{\rm A}^2$,

$$D_1 = D_T^{(\mathrm{H})} + \frac{1}{3}\eta + \frac{1}{6}\tau_0\,\overline{V}_{\rm A}^2, \tag{E17}$$

$$D_3 = \frac{1}{2}\,S\,\tau_0\left[\eta_T + \frac{49+42q}{105}\,\tau_0\,\overline{V}_{\rm A}^2\right], \tag{E18}$$

$$D_4 = \frac{1}{2}\,S\,\tau_0\left[\eta_T + \frac{145-2q}{105}\,\tau_0\,\overline{V}_{\rm A}^2\right]. \tag{E19}$$

Equation (E16) implies that $D_{yy}^{(\mathrm{H})} = D_{zz}^{(\mathrm{H})} = D_1$, $D_{xx}^{(\mathrm{H})} = D_1 - D_2$, $D_{xy}^{(\mathrm{H})} = D_3$, $D_{yx}^{(\mathrm{H})} = -D_4$, and other components of the total diffusion tensor $D_{ij}^{(\mathrm{H})}$ vanish. The turbulent magnetic helicity flux containing the source term $M_{ij}^{(\alpha)}\,\nabla_j \alpha_{\rm K}$ with $M_{ij}^{(\alpha)}$ being

$$M_{ij}^{(\alpha)} = \frac{1}{20q}\,\ell_0^2\,\overline{B}^2\left[(2q-1)\,\delta_{ij} + (20q-23)\,e_i^x\,e_j^x - \frac{16\,q\,(q-1)}{3q-1}\,S\,\tau_0\,e_i^x\,e_j^y\right]. \tag{E20}$$

The additional contribution $\boldsymbol{F}^{(\mathrm{S0})}$ to the turbulent magnetic helicity flux caused by the large-scale shear is given by

$$\boldsymbol{F}^{(\mathrm{S0})} = \frac{1}{3}\left[\frac{8q+35}{15}\,\ell_0^2\,\overline{B}^2 - \ell_b^2\,\langle \boldsymbol{b}^2\rangle\left(\frac{q-1}{q+1} - \frac{2\,(4q+1)}{15}\,\frac{\overline{V}_{\rm A}^2}{\langle \boldsymbol{u}^2\rangle}\right)\right]S\,\boldsymbol{e}^z. \tag{E21}$$

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